INFINITE GROUP PROBLEM CODE: FIGURES TEST SUITE

Abstract. The purpose of this file is to verify that changes to the code do not cause regressions in the plotting output. In particular, we verify that the code samples in figure captions of published papers can create the same (or improved) figures. This file should be visually inspected periodically.

1. Figures from *Light on the Infinite Group Relaxation*

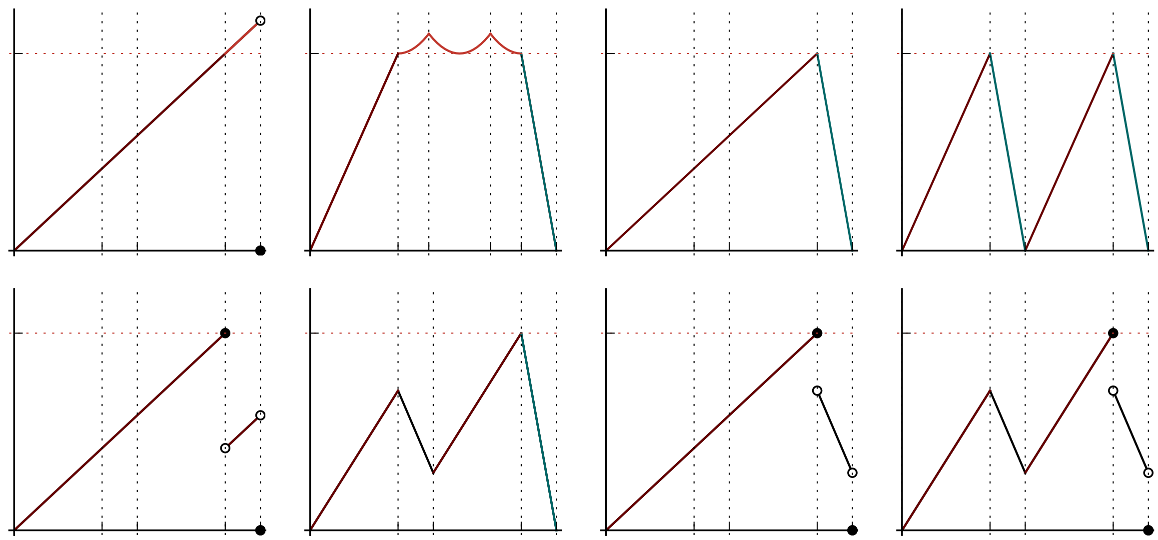


Figure 1. The hierarchy of valid, minimal, and extreme functions by example. . . Even without checking the dominance, it is easy to see that some functions cannot be minimal: they have some function values larger than 1 (*international orange*), but minimal valid functions are upper bounded by 1.

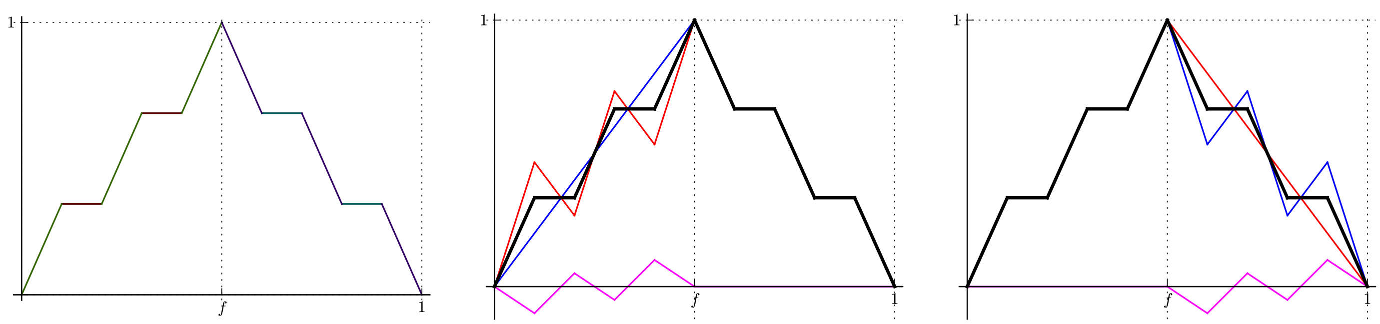


Figure 2. This function (`h = not_extreme_1()`) is minimal, but not extreme (and hence also not a facet), as proved by `extremality_test(h, show_plots=True)`. The procedure first shows that for any distinct minimal $\pi^1 = \pi + \bar{\pi}$ (*blue*), $\pi^2 = \pi - \bar{\pi}$ (*red*) such that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$, the functions π^1 and π^2 are continuous piecewise linear with the same breakpoints as π. A finite-dimensional extremality test then finds two linearly independent perturbations $\bar{\pi}$ (*magenta*), as shown.

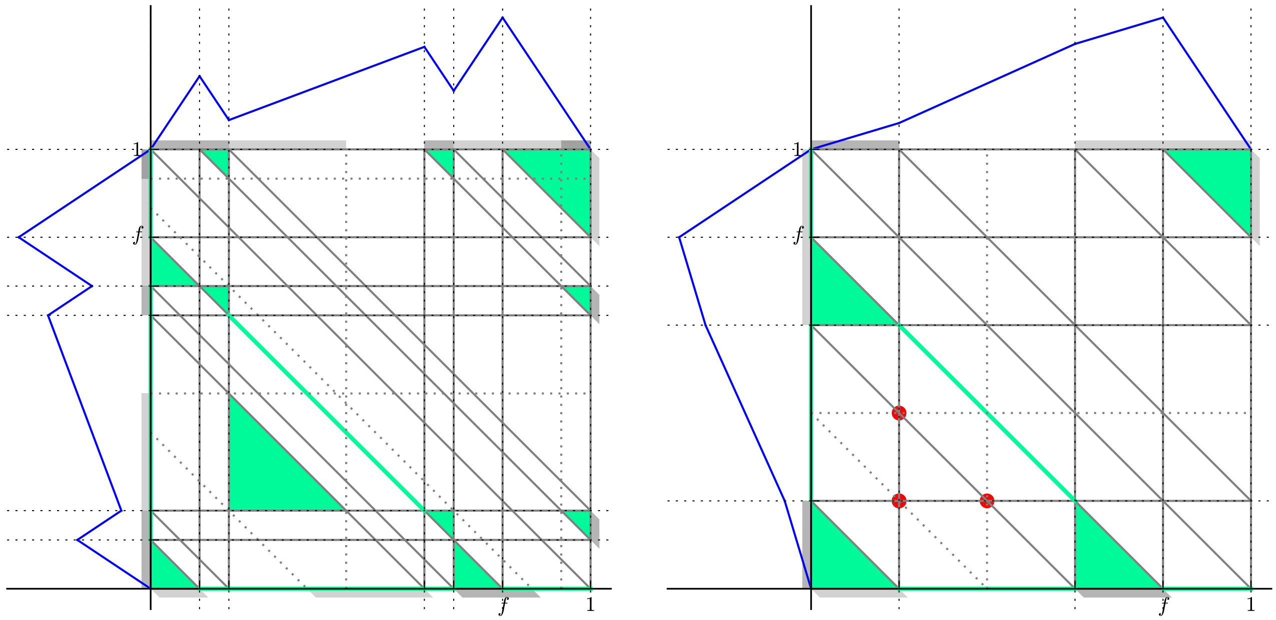


FIGURE 3. Two diagrams of a function (*blue graphs on the top and the left*) and its polyhedral complex $\Delta\mathcal{P}$ (*gray solid lines*), as plotted by the command `plot_2d_diagram(h)`. *Left*, `h = gj_forward_3_slope()` *(left)*. *Right*, `h = not_minimal_2()`. The set $E(\pi)$ in both cases is the union of the faces shaded in green. The *heavy diagonal green line* $x+y=f$ corresponds to the symmetry condition. Vertices of $\Delta\mathcal{P}$ do not necessarily project (*dotted gray lines*) to breakpoints. Vertices of the complex on which $\Delta\pi<0$ are shown as *red dots*. At the borders, the projections $p_i(F)$ of two-dimensional additive faces are shown as *gray shadows*: $p_1(F)$ at the top border, $p_2(F)$ at the left border, $p_3(F)$ at the bottom and the right borders.

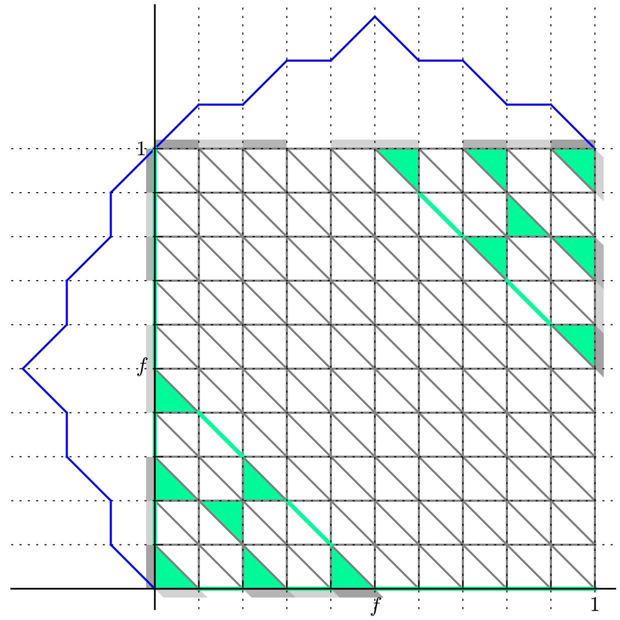


FIGURE 4. Diagram of a function (*blue graphs on the top and the left*) on the evenly spaced complex $\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ and the corresponding complex $\Delta\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ (*gray solid lines*), as plotted by the command `plot_2d_diagram(h)`, where `h = not_extreme_1()`. Faces of the complex on which $\Delta\pi=0$, i.e., additivity holds, are *shaded green*. The *heavy diagonal green lines* $x+y=f$ and $x+y=1+f$ correspond to the symmetry condition. At the borders, the projections $p_i(F)$ of two-dimensional additive faces are shown as *gray shadows*: $p_1(F)$ at the top border, $p_2(F)$ at the left border, $p_3(F)$ at the bottom and the right borders. Since the breakpoints of $\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ are equally spaced, also $\Delta\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ is very uniform, consisting only of points, lines, and triangles, and the projections are either a breakpoint in $\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ or an interval in $\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$; compare with Figure 3.

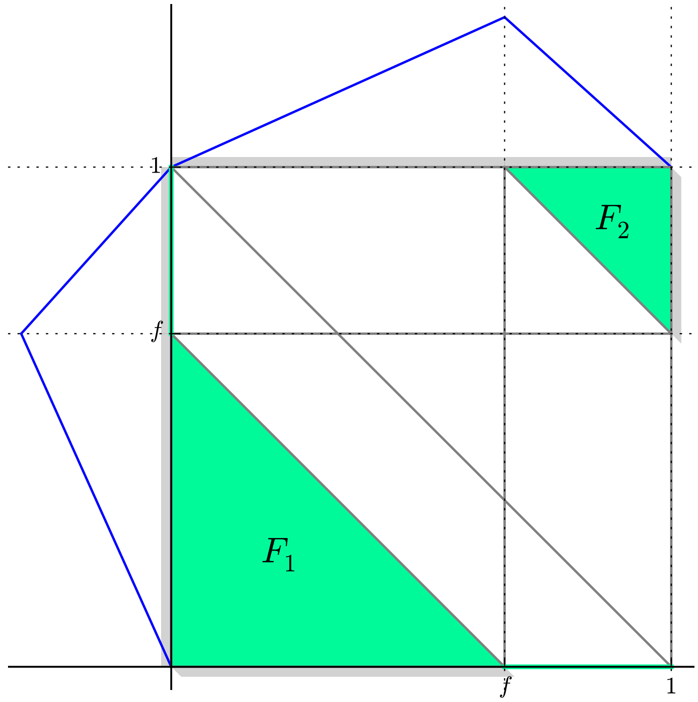


FIGURE 5. A diagram of a function of the type `gmic` (*blue graphs on the top and the left*) and its polyhedral complex $\Delta\mathcal{P}$ (*gray solid lines*), as plotted by the command `plot_2d_diagram(gmic(f=2/3))`. There are three combinatorial types of these diagrams, depending on whether $f < \frac{1}{2}$, $f = \frac{1}{2}$, or $f > \frac{1}{2}$. No matter what f is, the additivity domain $E(\pi)$ is the union of the faces $F_1 = F([0, f], [0, f], [0, f])$ and $F_2 = F([f, 1], [f, 1], [1 + f, 2])$, *shaded in green.* At the borders of each diagram, the projections $p_i(F)$ of two-dimensional additive faces are shown as *gray shadows*: $p_1(F)$ at the top border, $p_2(F)$ at the left border, $p_3(F)$ at the bottom and the right borders.

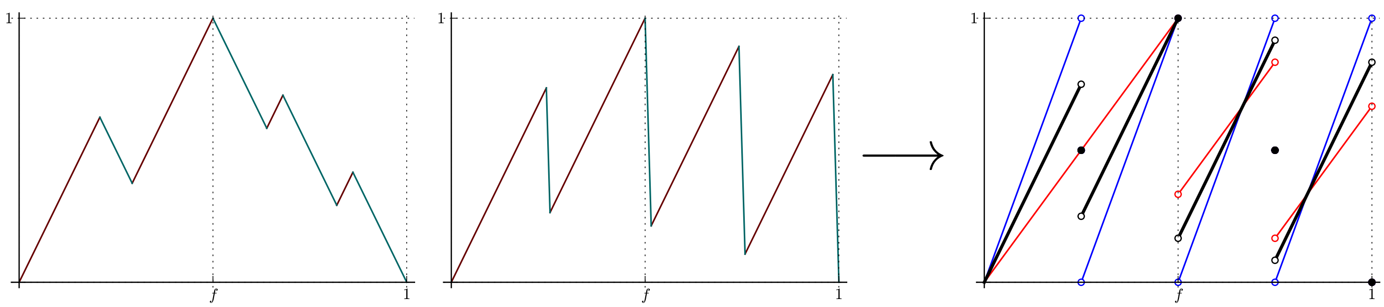


FIGURE 6. A pointwise limit of extreme functions that is not extreme. Consider the sequence of continuous extreme functions of type `gj_2_slope_repeat` set up for any $n \in \mathbb{Z}_+$ by `h = drlm_gj_2_slope_extreme_limit_to_nonextreme(n)`. For example, $n = 3$ *(left)* and $n = 50$ *(center).* This sequence converges to a non-extreme discontinuous minimal valid function, set up with `h = drlm_gj_2_slope_extreme_limit_to_nonextreme()` *(right).* The limit function π (*black*) is shown with two minimal functions π^1 (*blue*), π^2 (*red*) such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$.

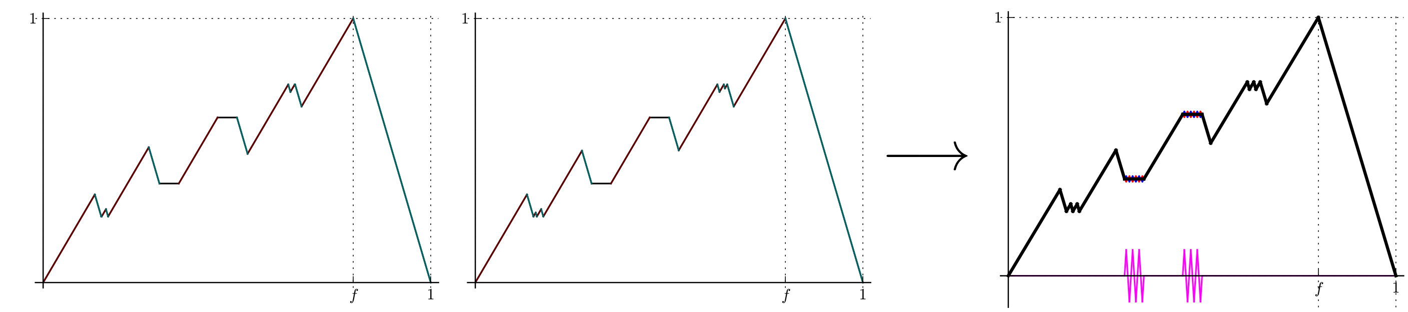


FIGURE 7. A uniform limit of extreme functions that is not extreme. The sequence of extreme functions of type `bhk_irrational`, set up with `h = bhk_irrational_extreme_limit_to_rational_nonextreme(n)` where $n = 1$ *(left)*, $n = 2$ *(center)*, … converges to a non-extreme function, set up with `h = bhk_irrational_extreme_limit_to_rational_nonextreme()` *(right)*. The limit function π (*black*) is shown with two minimal functions π^1 (*blue*), π^2 (*red*) such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$ and a scaling of the perturbation function $\bar{\pi} = \pi^1 - \pi$ (*magenta*).

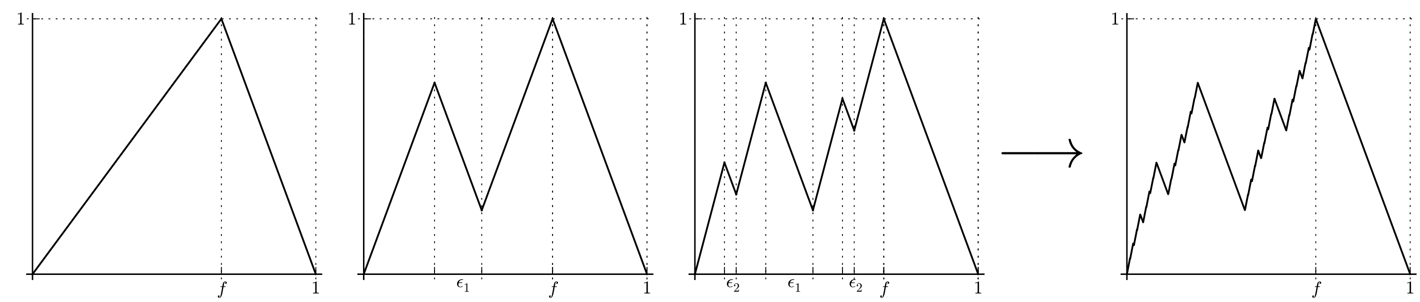


FIGURE 8. First steps (ψ_0 = `gmic()`, ψ_1, ψ_2) in the construction of the continuous non–piecewise linear limit function ψ = `bccz_counterexample()`.

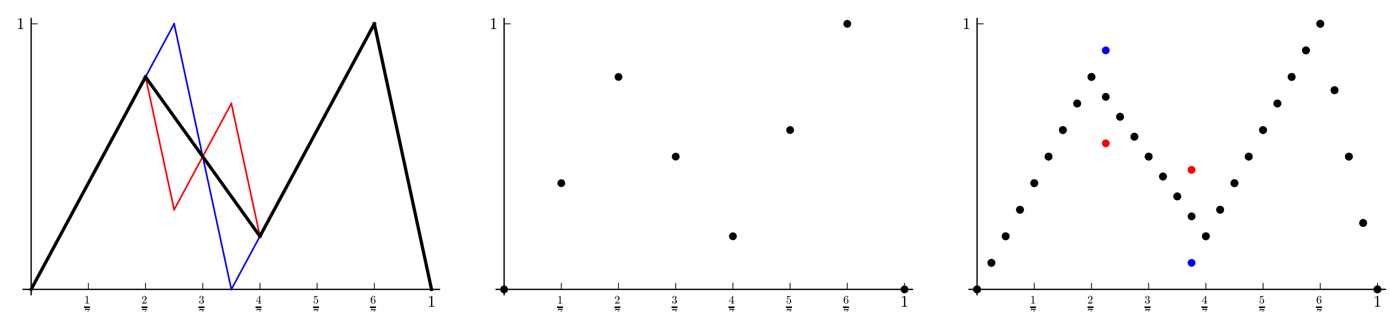
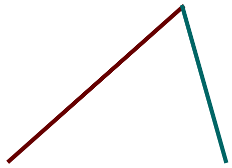
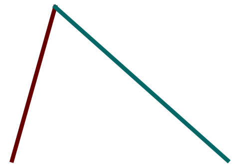
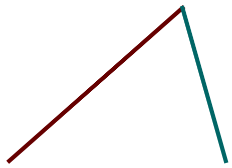
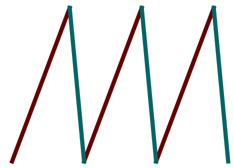
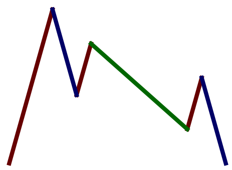
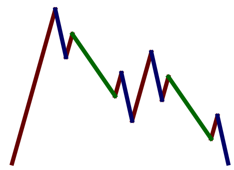
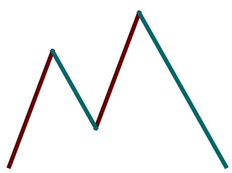
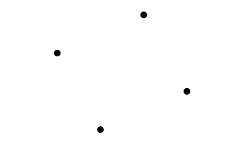
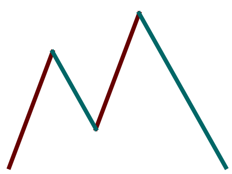
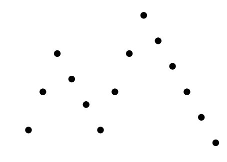
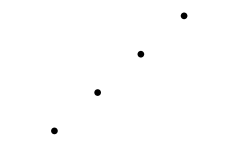
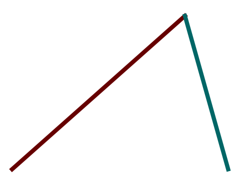
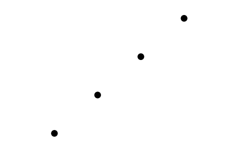
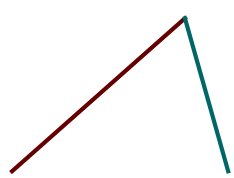


FIGURE 9. This function (`h = drlm_not_extreme_1()`) is minimal, but not extreme (and hence also not a facet), as proved by `extremality_test(h, show_plots=True)` by demonstrating a perturbation. The red and blue perturbations describe the minimal functions π^1, π^2 that verify that π is not extreme. These minimal functions necessarily have more breakpoints than π. This is because $\pi|_{\frac{1}{q}\mathbb{Z}}$ with $q = 7$, as depicted in the middle figure, is extreme for the finite group problem $R_f(\frac{1}{q}\mathbb{Z}, \mathbb{Z})$. However, $\pi|_{\frac{1}{2q}\mathbb{Z}}$ is not extreme for $R_f(\frac{1}{2q}\mathbb{Z}, \mathbb{Z})$. The discrete perturbations, depicted on the right, are interpolated to obtain the continuous functions π^1, π^2.

TABLE 1. An updated compendium of known extreme functions for the infinite group problem V. Procedures.

	Graphs		
Procedure[a]	From	To	Notes
`automorphism`			From Johnson
`multiplicative_homomorphism`			
`projected_sequential_merge`			Operation $\Diamond_n^1$ from Dey–Richard
`restrict_to_finite_group`			Restrictions to finite group problems $R_f(\frac{1}{q}\mathbb{Z}, \mathbb{Z})$ preserve extremality if f and all breakpoints lie in $\frac{1}{q}\mathbb{Z}$.
`restrict_to_finite_group` `(oversampling=3)`			If `oversampling` by a factor $m \geq 3$, the restriction is extreme for $R_f(\frac{1}{mq}\mathbb{Z}, \mathbb{Z})$ if and only if the original function is extreme.
`interpolate_to_infinite_group`			Interpolation from finite group problems $R_f(\frac{1}{q}\mathbb{Z}, \mathbb{Z})$ preserves minimality, but in general not extremality.
`two_slope_fill_in`			Described by Gomory–Johnson, Johnson. For $k = 1$, if minimal, equal to `interpolate_to_infinite_group` (above).

[a]A procedure name shown in typewriter font is the name of the corresponding function in the accompanying Sage program.