INFINITE GROUP PROBLEM CODE:
FIGURES TEST SUITE

ABSTRACT. The purpose of this file is to verify that changes to the code do not cause regressions in the plotting output. In particular, we verify that the code samples in figure captions of published papers can create the same (or improved) figures. This file should be visually inspected periodically.

1. FIGURES FROM *Light on the Infinite Group Relaxation*

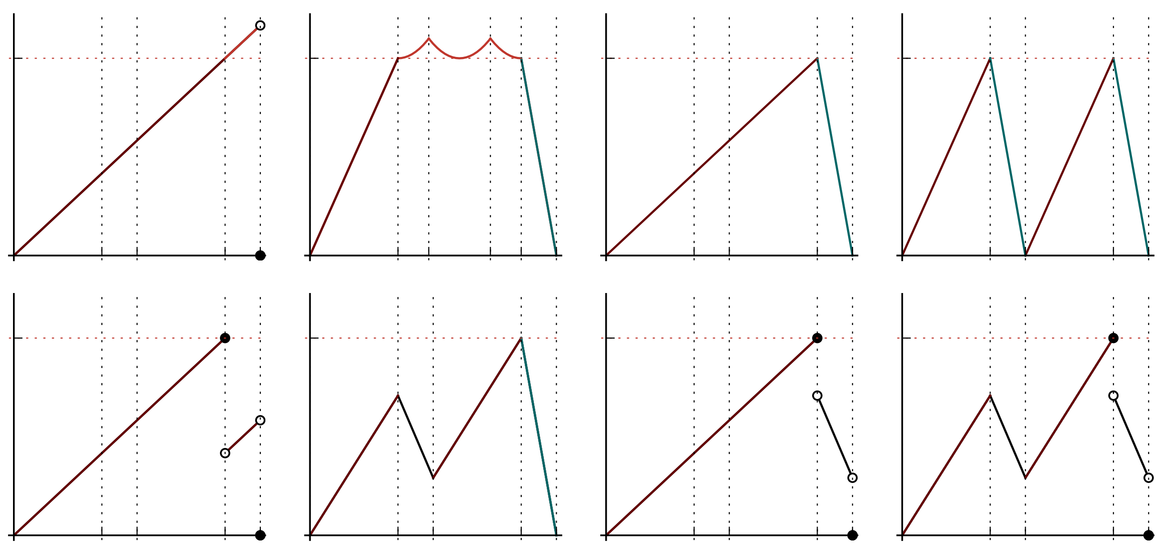


FIGURE 1. The hierarchy of valid, minimal, and extreme functions by example... Even without checking the dominance, it is easy to see that some functions cannot be minimal: they have some function values larger than 1 (*international orange*), but minimal valid functions are upper bounded by 1.

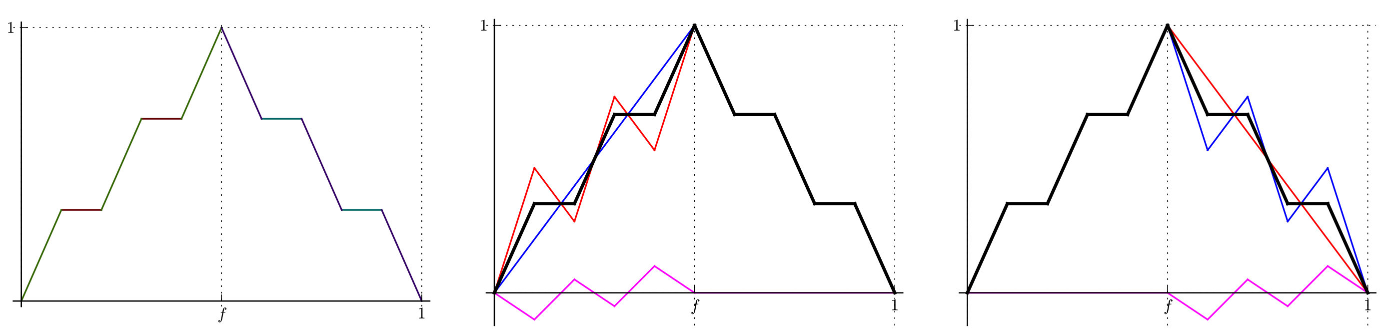


FIGURE 2. This function (`h = not_extreme_1()`) is minimal, but not extreme (and hence also not a facet), as proved by `extremality_test(h, show_plots=True)`. The procedure first shows that for any distinct minimal $\pi^1 = \pi + \bar{\pi}$ (*blue*), $\pi^2 = \pi - \bar{\pi}$ (*red*) such that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$, the functions π^1 and π^2 are continuous piecewise linear with the same breakpoints as π. A finite-dimensional extremality test then finds two linearly independent perturbations $\bar{\pi}$ (*magenta*), as shown.

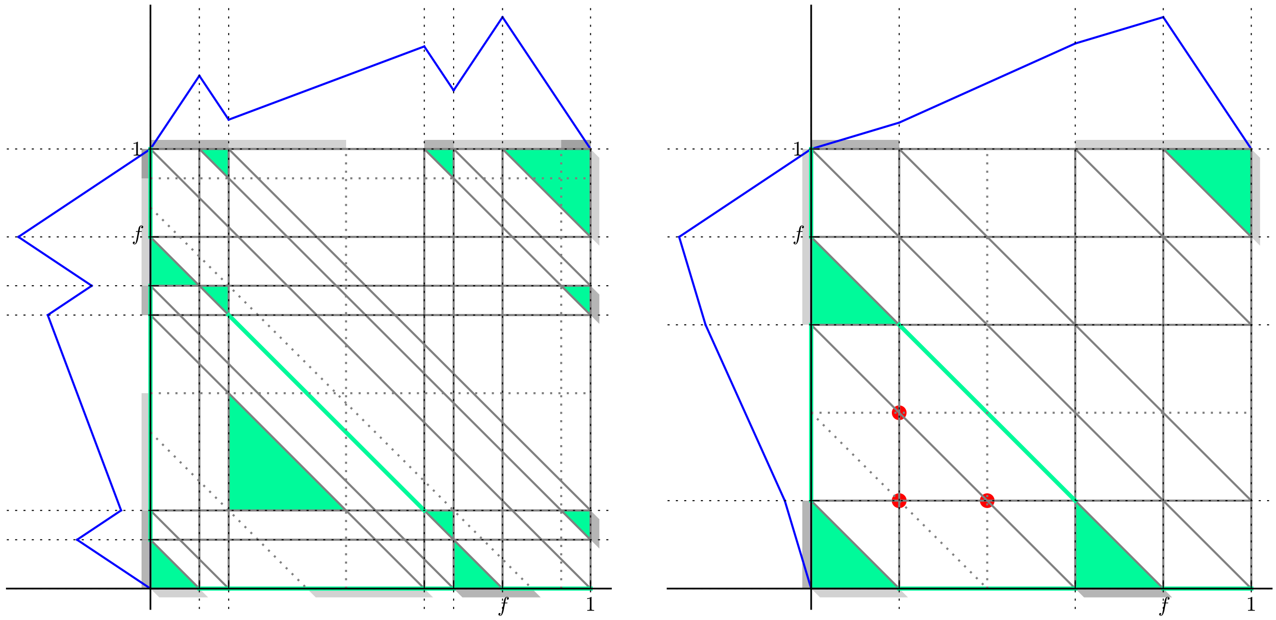


Figure 3. Two diagrams of a function (*blue graphs on the top and the left*) and its polyhedral complex $\Delta\mathcal{P}$ (*gray solid lines*), as plotted by the command `plot_2d_diagram(h)`. *Left*, `h = gj_forward_3_slope()` *(left)*. *Right*, `h = not_minimal_2()`. The set $E(\pi)$ in both cases is the union of the faces shaded in green. The *heavy diagonal green line* $x + y = f$ corresponds to the symmetry condition. Vertices of $\Delta\mathcal{P}$ do not necessarily project (*dotted gray lines*) to breakpoints. Vertices of the complex on which $\Delta\pi < 0$ are shown as *red dots*. At the borders, the projections $p_i(F)$ of two-dimensional additive faces are shown as *gray shadows*: $p_1(F)$ at the top border, $p_2(F)$ at the left border, $p_3(F)$ at the bottom and the right borders.

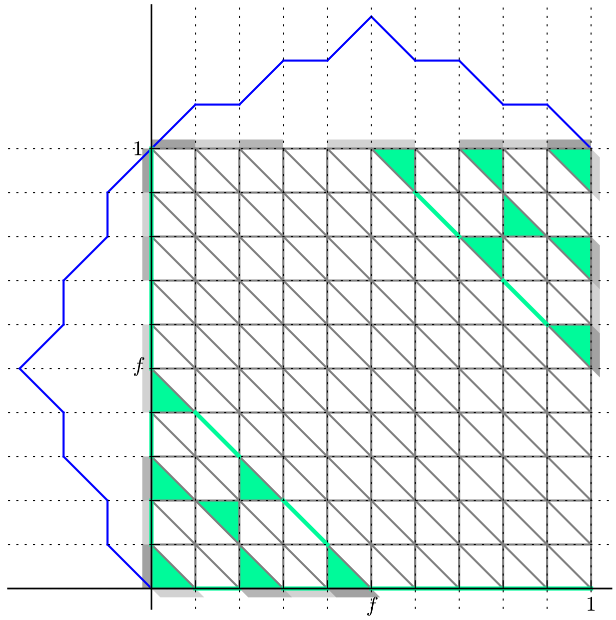


Figure 4. Diagram of a function (*blue graphs on the top and the left*) on the evenly spaced complex $\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ and the corresponding complex $\Delta\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ (*gray solid lines*), as plotted by the command `plot_2d_diagram(h)`, where `h = not_extreme_1()`. Faces of the complex on which $\Delta\pi = 0$, i.e., additivity holds, are *shaded green*. The *heavy diagonal green lines* $x + y = f$ and $x + y = 1 + f$ correspond to the symmetry condition. At the borders, the projections $p_i(F)$ of two-dimensional additive faces are shown as *gray shadows*: $p_1(F)$ at the top border, $p_2(F)$ at the left border, $p_3(F)$ at the bottom and the right borders. Since the breakpoints of $\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ are equally spaced, also $\Delta\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ is very uniform, consisting only of points, lines, and triangles, and the projections are either a breakpoint in $\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ or an interval in $\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$; compare with Figure 3.

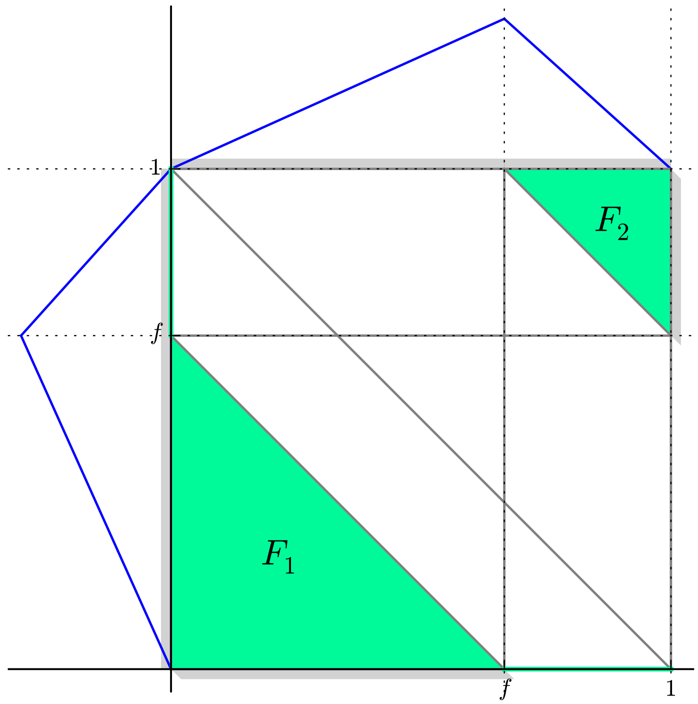


FIGURE 5. A diagram of a function of the type `gmic` (*blue graphs on the top and the left*) and its polyhedral complex $\Delta\mathcal{P}$ (*gray solid lines*), as plotted by the command `plot_2d_diagram(gmic(f=2/3))`. There are three combinatorial types of these diagrams, depending on whether $f < \frac{1}{2}$, $f = \frac{1}{2}$, or $f > \frac{1}{2}$. No matter what f is, the additivity domain $E(\pi)$ is the union of the faces $F_1 = F([0, f], [0, f], [0, f])$ and $F_2 = F([f, 1], [f, 1], [1 + f, 2])$, *shaded in green*. At the borders of each diagram, the projections $p_i(F)$ of two-dimensional additive faces are shown as *gray shadows*: $p_1(F)$ at the top border, $p_2(F)$ at the left border, $p_3(F)$ at the bottom and the right borders.

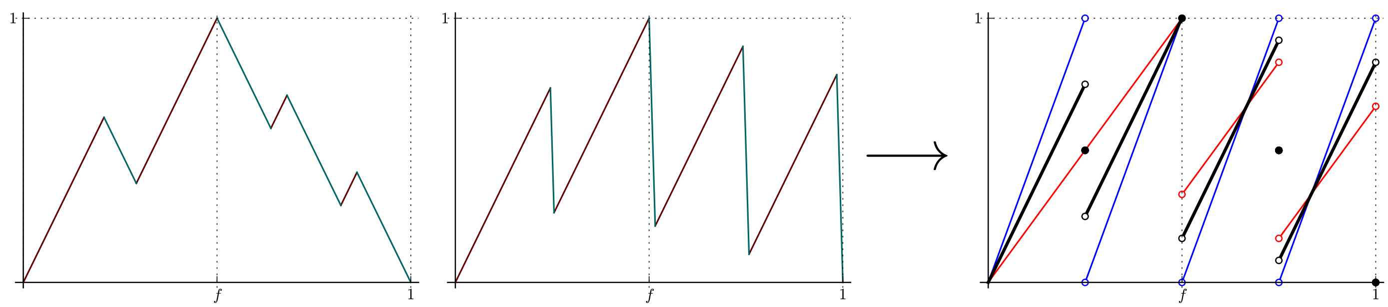


FIGURE 6. A pointwise limit of extreme functions that is not extreme. Consider the sequence of continuous extreme functions of type `gj_2_slope_repeat` set up for any $n \in \mathbb{Z}_+$ by `h = drlm_gj_2_slope_extreme_limit_to_nonextreme(n)`. For example, $n = 3$ *(left)* and $n = 50$ *(center)*. This sequence converges to a non-extreme discontinuous minimal valid function, set up with `h = drlm_gj_2_slope_extreme_limit_to_nonextreme()` *(right)*. The limit function π (*black*) is shown with two minimal functions π^1 (*blue*), π^2 (*red*) such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$.

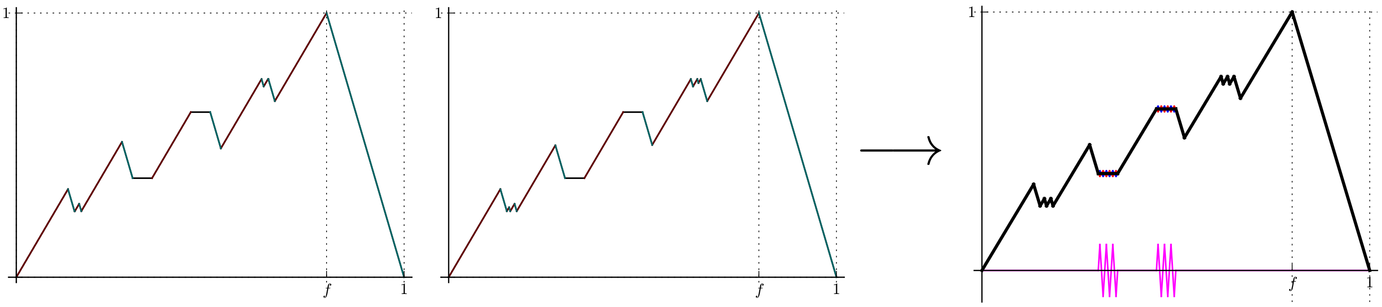


FIGURE 7. A uniform limit of extreme functions that is not extreme. The sequence of extreme functions of type `bhk_irrational`, set up with `h = bhk_irrational_extreme_limit_to_rational_nonextreme(n)` where $n = 1$ *(left)*, $n = 2$ *(center)*, … converges to a non-extreme function, set up with `h = bhk_irrational_extreme_limit_to_rational_nonextreme()` *(right)*. The limit function π (*black*) is shown with two minimal functions π^1 (*blue*), π^2 (*red*) such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$ and a scaling of the perturbation function $\bar{\pi} = \pi^1 - \pi$ (*magenta*).

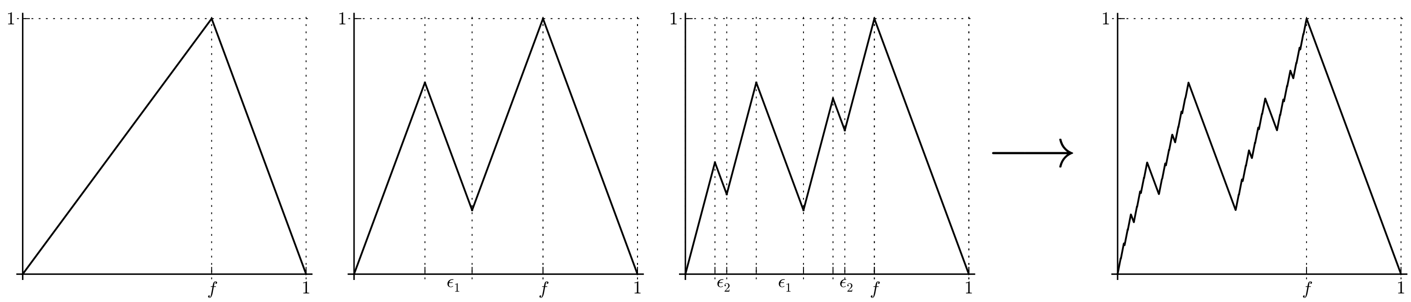


FIGURE 8. First steps (ψ_0 = `gmic()`, ψ_1, ψ_2) in the construction of the continuous non–piecewise linear limit function ψ = `bccz_counterexample()`.

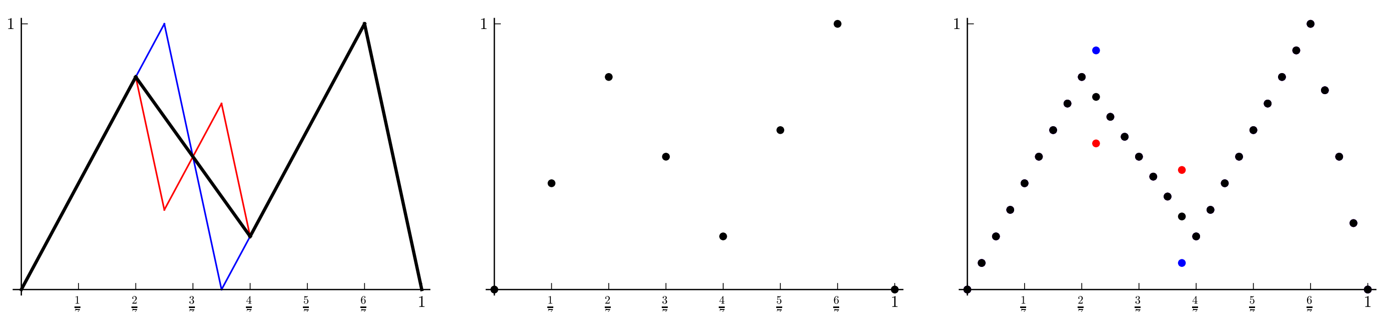
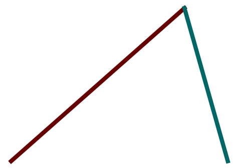
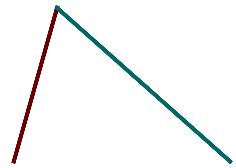
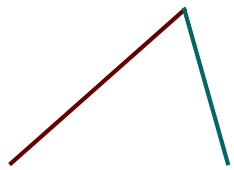
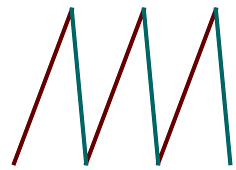
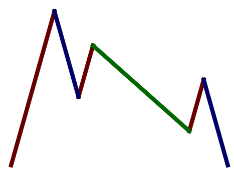
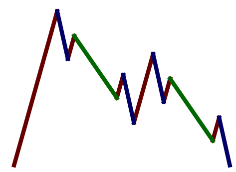
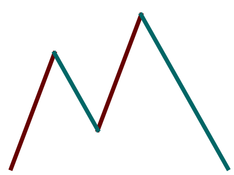
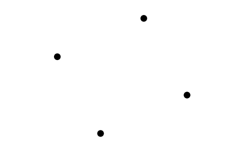
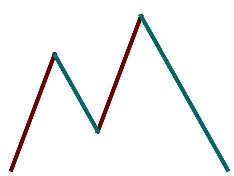
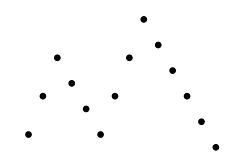
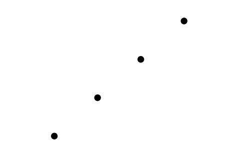
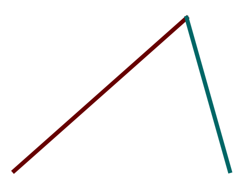
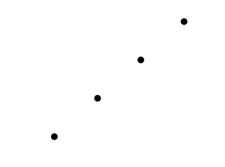
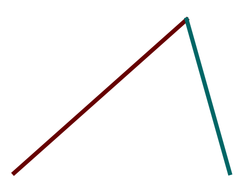


FIGURE 9. This function (`h = drlm_not_extreme_1()`) is minimal, but not extreme (and hence also not a facet), as proved by `extremality_test(h, show_plots=True)` by demonstrating a perturbation. The red and blue perturbations describe the minimal functions π^1, π^2 that verify that π is not extreme. These minimal functions necessarily have more breakpoints than π. This is because $\pi|_{\frac{1}{q}\mathbb{Z}}$ with $q = 7$, as depicted in the middle figure, is extreme for the finite group problem $R_f(\frac{1}{q}\mathbb{Z}, \mathbb{Z})$. However, $\pi|_{\frac{1}{2q}\mathbb{Z}}$ is not extreme for $R_f(\frac{1}{2q}\mathbb{Z}, \mathbb{Z})$. The discrete perturbations, depicted on the right, are interpolated to obtain the continuous functions π^1, π^2.

TABLE 1. An updated compendium of known extreme functions for the infinite group problem V. Procedures.

	Graphs		
Procedure[a]	From	To	Notes
`automorphism`			From Johnson
`multiplicative_homomorphism`			
`projected_sequential_merge`			Operation $\diamondsuit_n^1$ from Dey–Richard
`restrict_to_finite_group`			Restrictions to finite group problems $R_f(\frac{1}{q}\mathbb{Z},\mathbb{Z})$ preserve extremality if f and all breakpoints lie in $\frac{1}{q}\mathbb{Z}$.
`restrict_to_finite_group` `(oversampling=3)`			If `oversampling` by a factor $m \geq 3$, the restriction is extreme for $R_f(\frac{1}{mq}\mathbb{Z},\mathbb{Z})$ if and only if the original function is extreme.
`interpolate_to_infinite_group`			Interpolation from finite group problems $R_f(\frac{1}{q}\mathbb{Z},\mathbb{Z})$ preserves minimality, but in general not extremality.
`two_slope_fill_in`			Described by Gomory–Johnson, Johnson. For $k = 1$, if minimal, equal to `interpolate_to_infinite_group` (above).

[a]A procedure name shown in typewriter font is the name of the corresponding function in the accompanying Sage program.

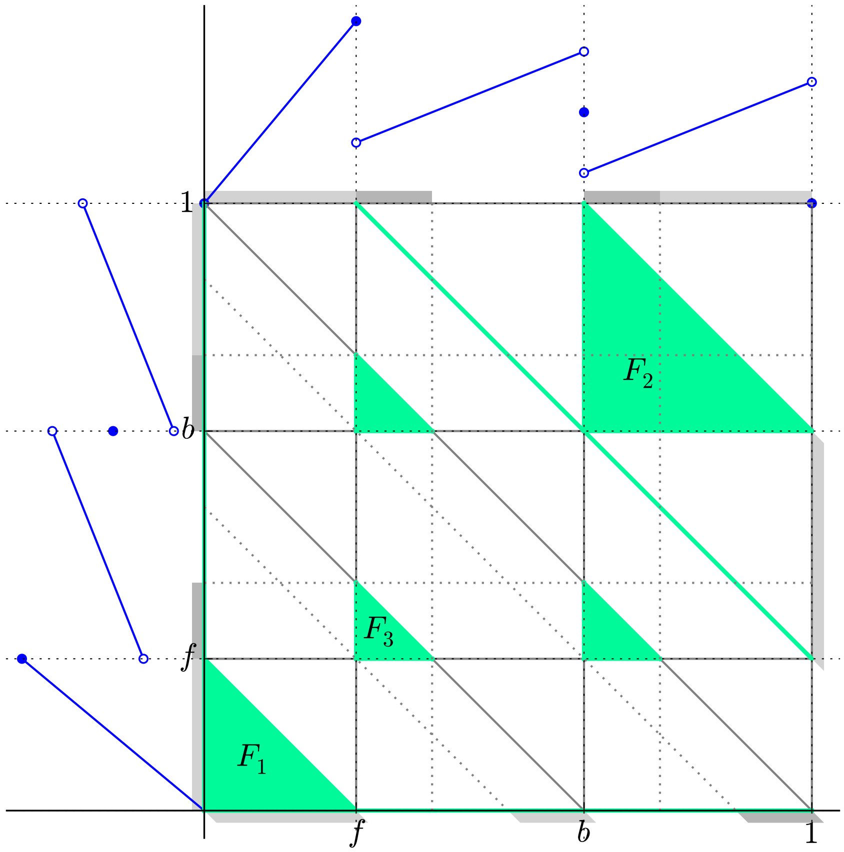
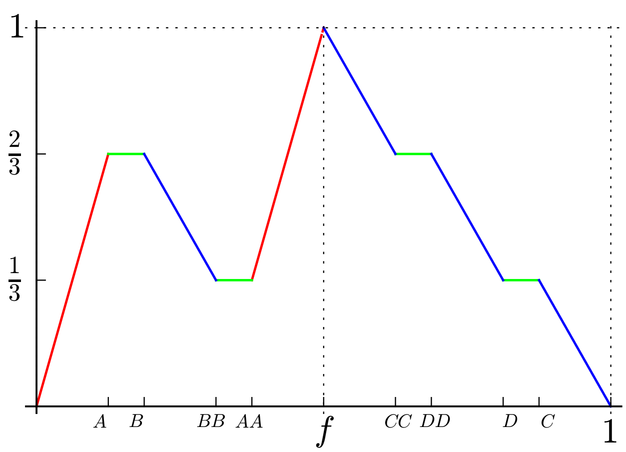


FIGURE 1. Diagram of the function `rlm_dpl1_extreme_3a` (*blue graphs on the top and the left*) and its polyhedral complex $\Delta\mathcal{P}$ (*gray solid lines*). The set $E(\pi)$ is the union of the faces shaded in green. The *heavy diagonal green line* $x + y = 1 + f$ corresponds to the symmetry condition (the line $x + y = f$ appears as an edge of F_1). Vertices of $\Delta\mathcal{P}$ do not necessarily project (*dotted gray lines*) to breakpoints. At the borders, the projections $p_i(F)$ of two-dimensional additive faces are shown as *gray shadows*: $p_1(F)$ at the top border, $p_2(F)$ at the left border, $p_3(F)$ at the bottom and the right borders.



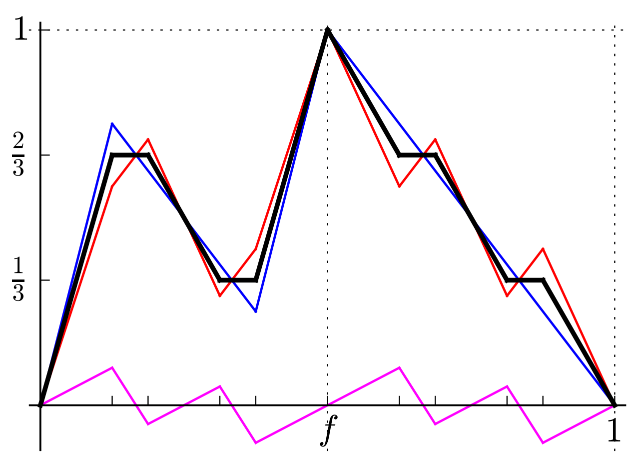


FIGURE 2. The function `chen_3_slope_not_extreme` is minimal, but not extreme, as proved by `extremality_test(h, show_plots=True)`. The procedure first shows that for any distinct minimal $\pi^1 = \pi + \bar{\pi}$ (*blue*), $\pi^2 = \pi - \bar{\pi}$ (*red*) such that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$, the functions π^1 and π^2 are continuous piecewise linear with the same breakpoints as π. A finite-dimensional extremality test then finds a perturbation $\bar{\pi}$ (*magenta*), as shown.

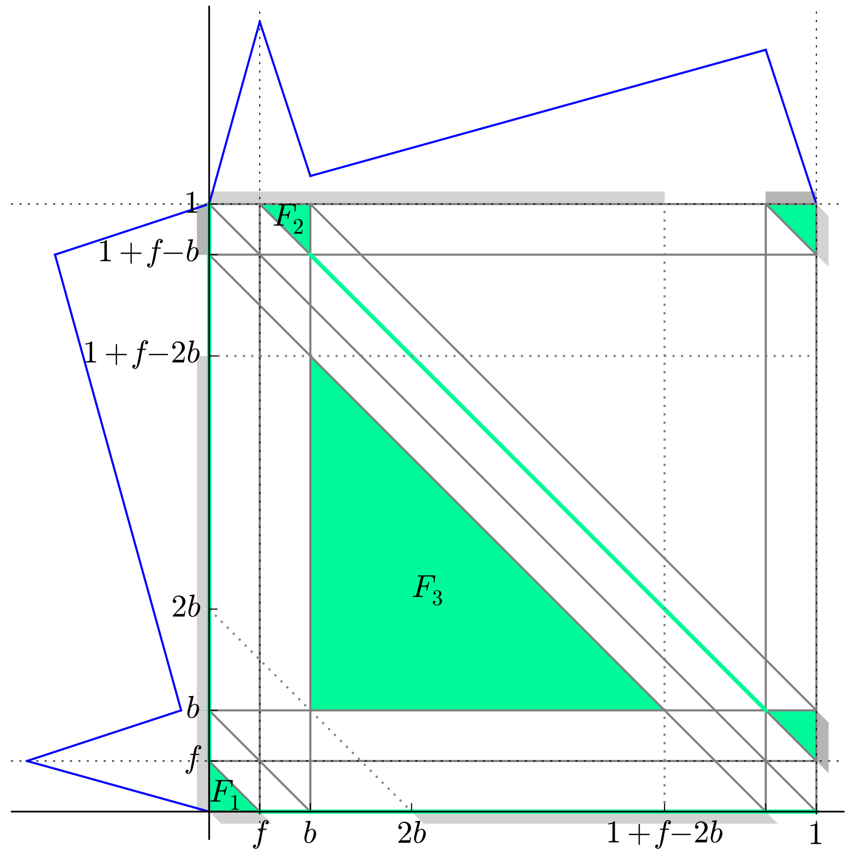


FIGURE 3. The `drlm_backward_3_slope` function

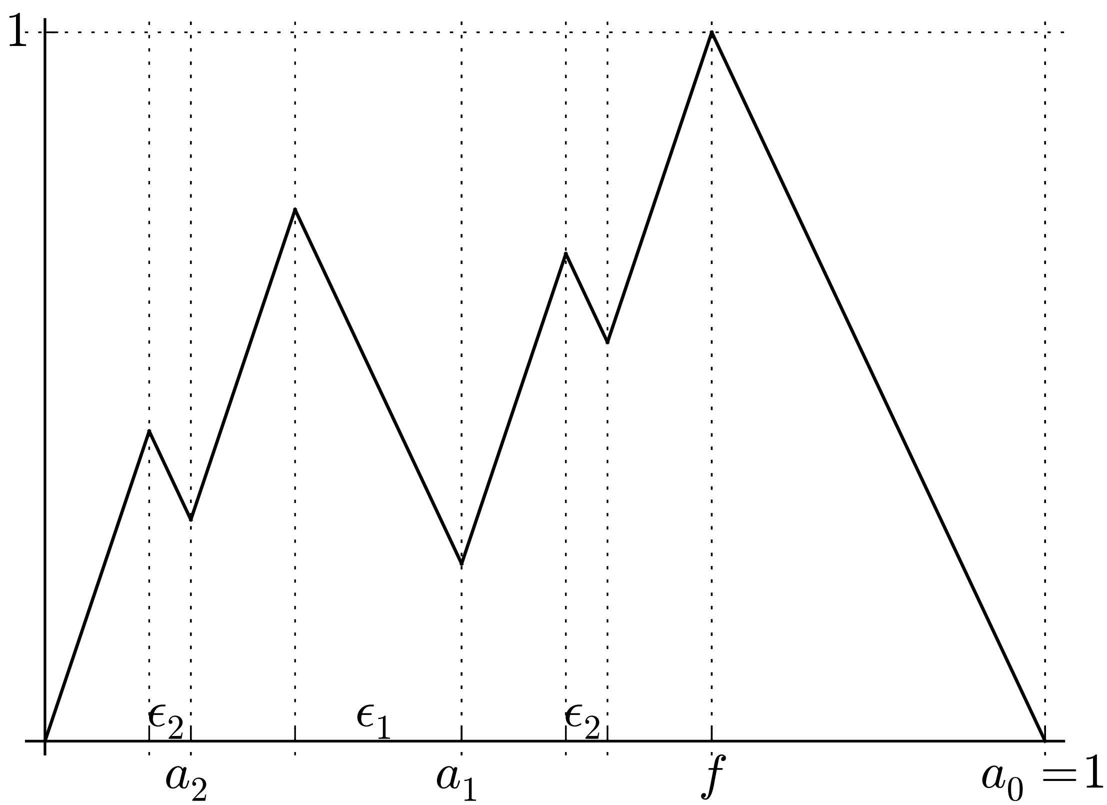
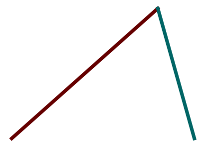


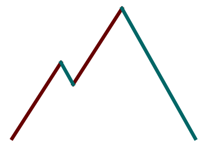
FIGURE 4. The `kf_n_step_mir` function

3. Electronic compendium

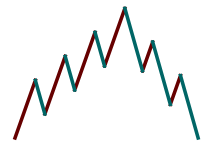
Table 1. An overview of the Electronic Compendium of extreme functions, available at `https://github.com/mkoeppe/infinite-group-relaxation-code`



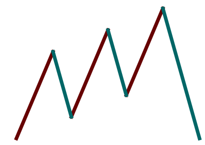
`gmic`



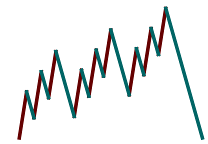
`gj_2_slope`



`gj_2_slope_repeat`

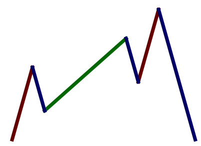


`dg_2_step_mir`

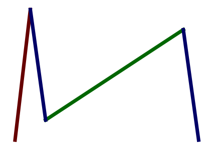


`kf_n_step_mir`

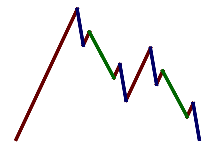
`bccz_counterexample`



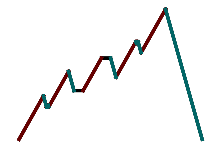
`gj_forward_3_slope`



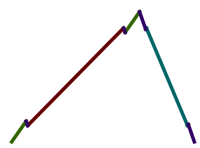
`drlm_backward_3_slope`



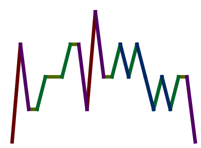
`dr_projected_sequential_merge_3_slope`



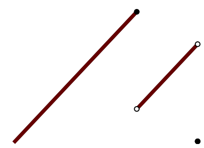
`bhk_irrational`



`chen_4_slope`



`hildebrand_5_slope_22_1`



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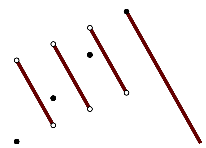

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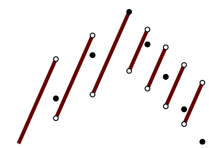
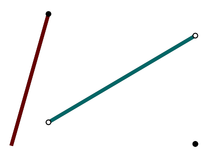
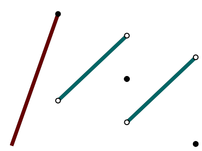


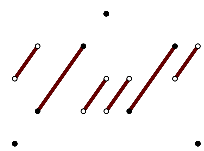
`drlm_2_slope_limit`



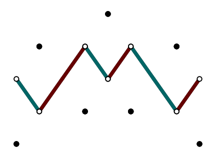
`drlm_3_slope_limit`



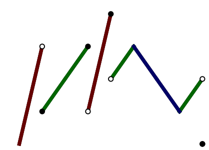
`rlm_dpl1_extreme_3a`



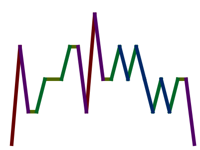
`hildebrand_2_sided_discont_1_slope_1`



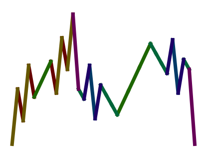
`hildebrand_2_sided_discont_2_slope_1`



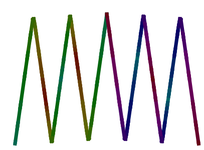
`hildebrand_discont_3_slope_1`



`hildebrand_5_slope_22_1`



`kzh_7_slope_1`



`kzh_28_slope_1`

4. Figures from *New computer-based search strategies for extreme functions of the Gomory–Johnson infinite group problem*

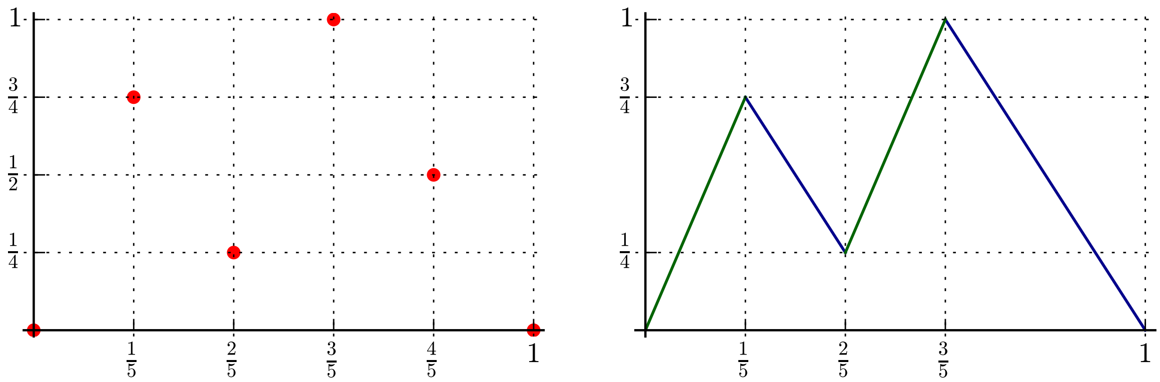


Figure 1. The 2-slope extreme function `gj_2_slope`, discovered by Gomory and Johnson. *Left*, `gj_2_slope` for the finite group problem with $q = 5$ and $f = \frac{3}{5}$, obtained by `restrict_to_finite_group(gj_2_slope())`. It is a discrete function whose interpolation is the right subfigure. *Right*, `gj_2_slope` for the infinite group problem with $f = \frac{3}{5}$. It is a continuous piecewise linear function with two slopes, although it has four pieces. Its restriction to $\frac{1}{5}\mathbb{Z}$ is the left subfigure.

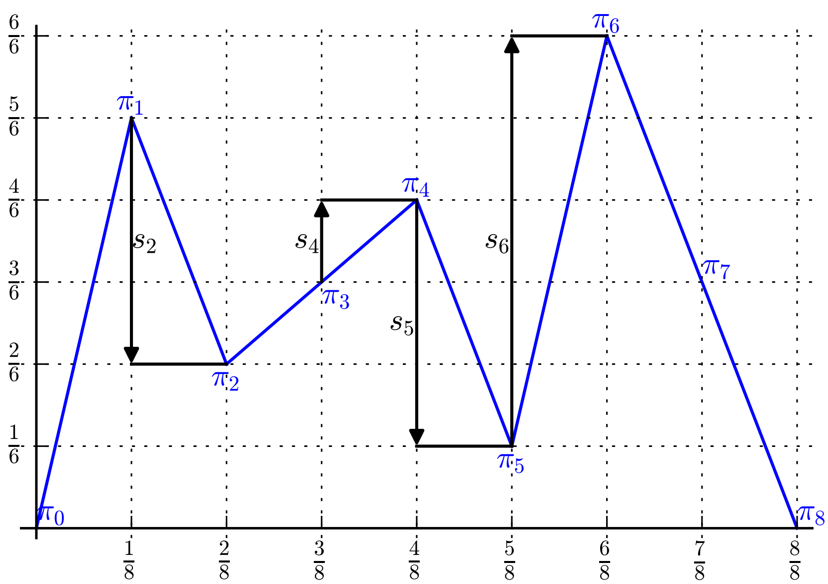


Figure 2. The $q \times v$ grid discretization of the space of continuous piecewise linear functions with rational data. Here $q = 8$ and $v = 6$.

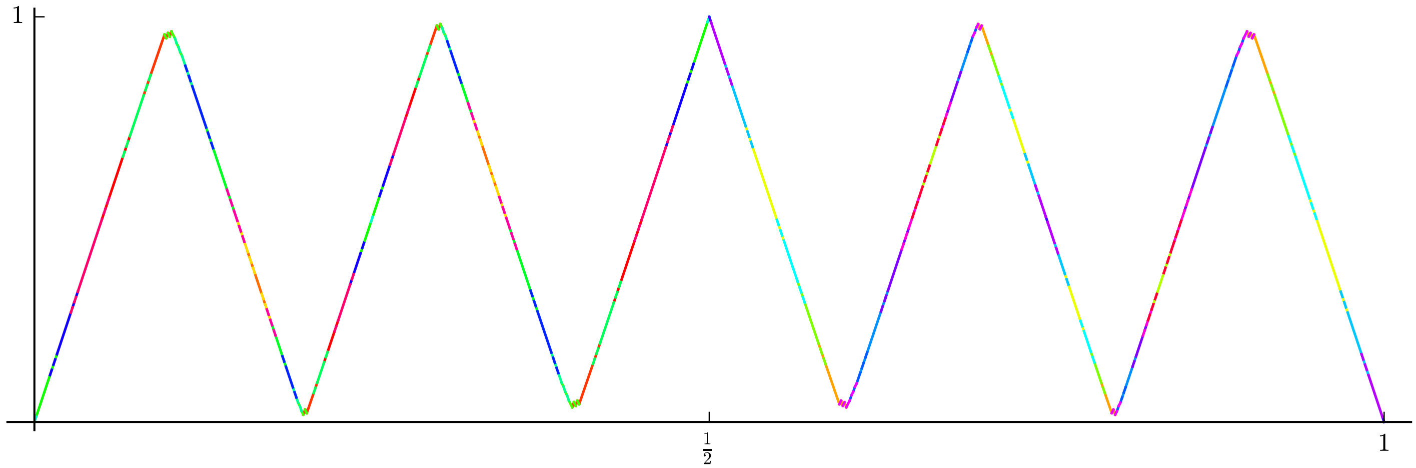


FIGURE 3. A 28-slope extreme function `kzh_28_slope_1` found by our search code. Each color in the plotting corresponds to a different slope value.

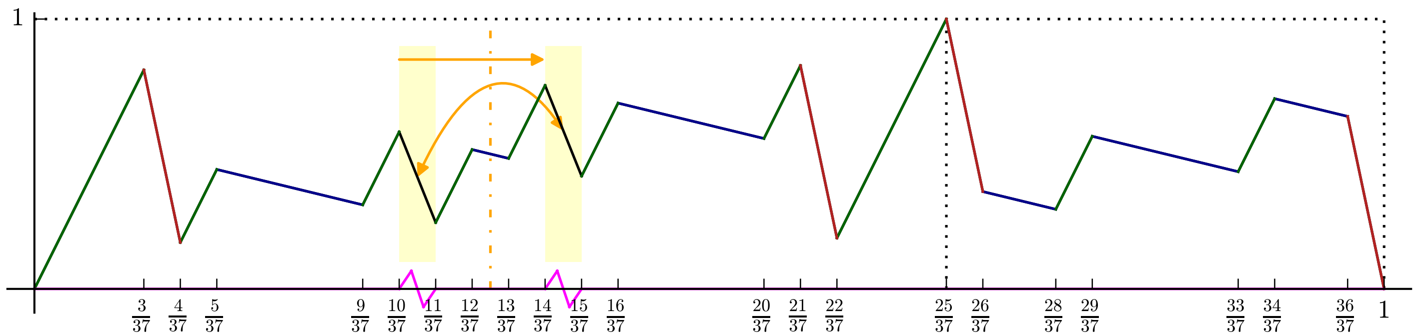
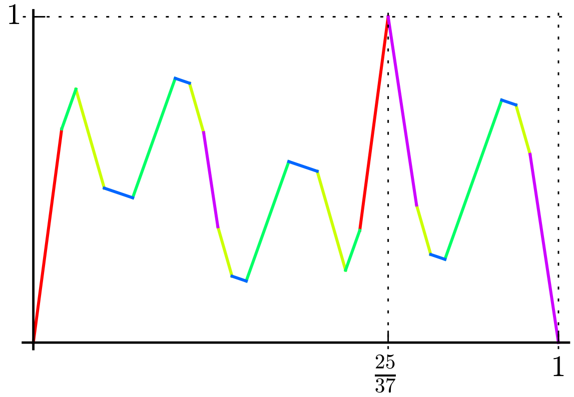


FIGURE 4. The example `kzh_2q_example_1`, showing that an oversampling factor of $m = 3$ is best possible.



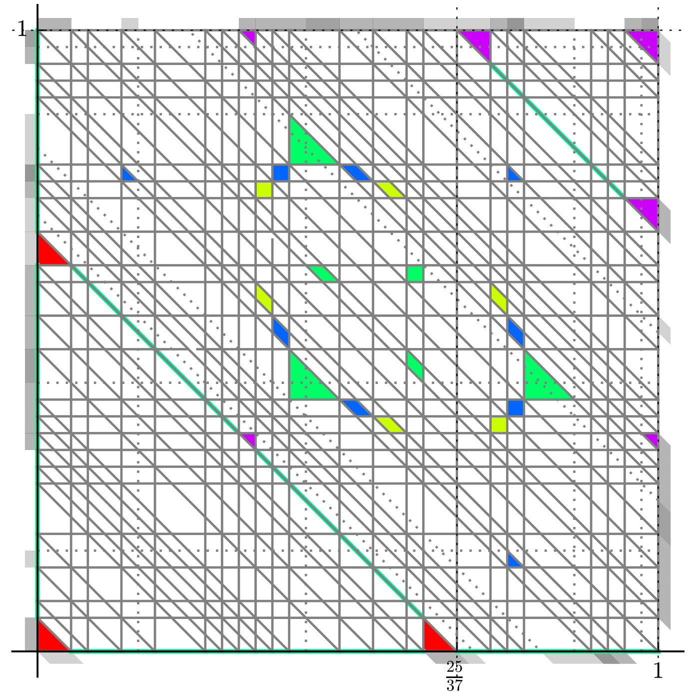
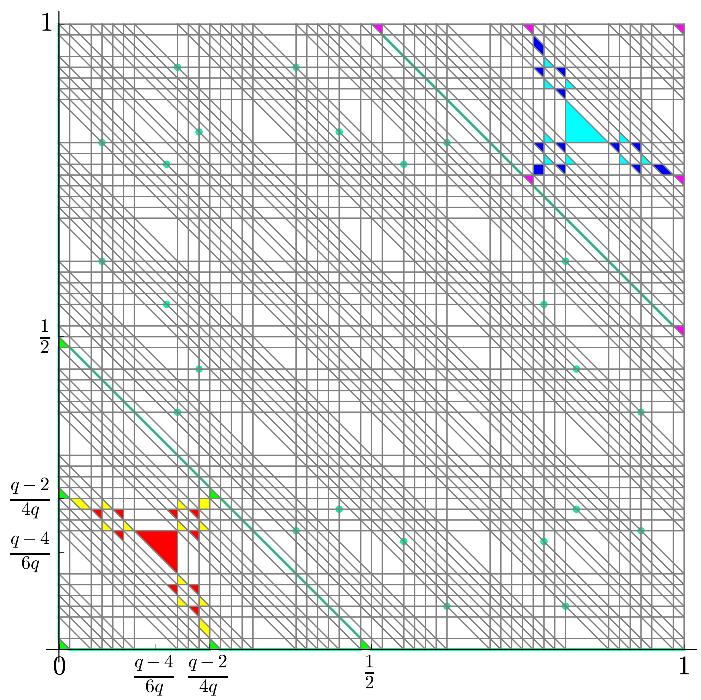


FIGURE 9. The 5-slope extreme function `kzh_5_slope_fulldim_1` found by our search code (*left*). Its two-dimensional polyhedral complex $\Delta\mathcal{P}$ (*right*), as plotted by the command `plot_2d_diagram(h,colorful=True)`, does not have any lower-dimensional maximal additive faces except for the symmetry reflection or $x = 0$ or $y = 0$.



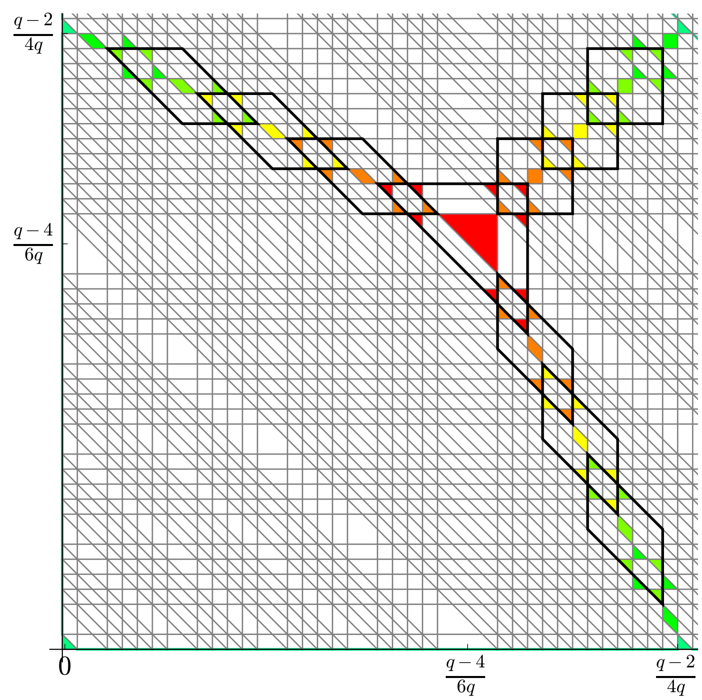


FIGURE 11. Special patterns on the two-dimensional polyhedral complex $\Delta\mathcal{P}_{\frac{1}{q}\mathbb{Z}}$. *Left,* the $\Delta\mathcal{P}_{\frac{1}{q}\mathbb{Z}}$ of the 6-slope extreme function `kzh_6_slope_1` with $q = 58$. We observe that the additive triangles are located in the lower left and upper right corners. The function has the same slopes on the intervals that are projections of the same color additive triangles. The 6-pointed star patterns appear several times. *Right,* the lower-left corner of $\Delta\mathcal{P}_{\frac{1}{q}\mathbb{Z}}$ of the 10-slope extreme function `kzh_10_slope_1` with $q = 166$, where we see that the 6-pointed stars are actually the result of additivity patterns within certain intersecting quadrilaterals (*black*), which connect like links of three chains.

5. Figures from *Equivariant Perturbation in Gomory and Johnson's Infinite Group Problem. V. Software for the continuous and discontinuous 1-row case*

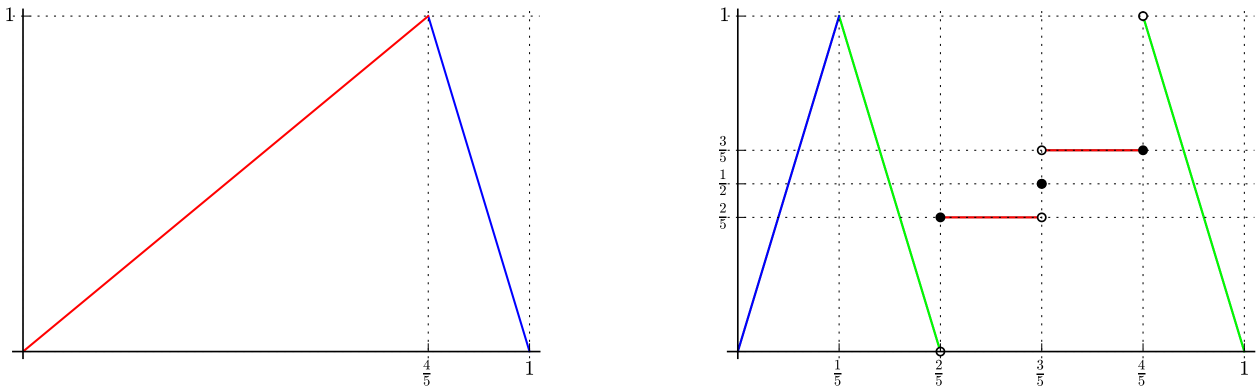


Figure 1. Two piecewise linear functions, as plotted by the command `plot_with_colored_slopes(h)`. *Left,* continuous extreme function `h = gmic()`. *Right,* random discontinuous function `h = equiv5_random_discont_1()`, generated by `random_piecewise_function(xgrid=5, ygrid=5, continuous_proba=1/3, symmetry=True)`.

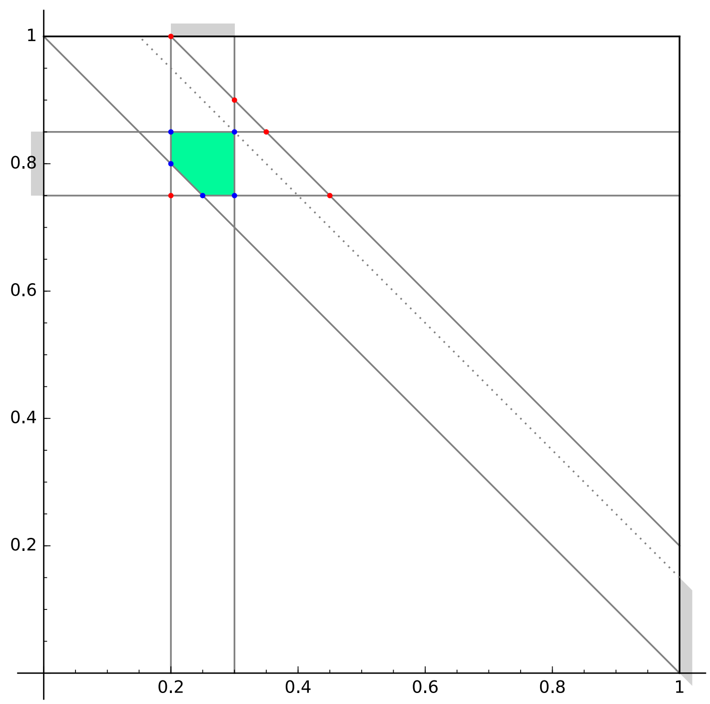


Figure 2. An example of a face $F = F(I, J, K)$ of the 2-dimensional polyhedral complex $\Delta\mathcal{P}$, set up by `F = Face([[0.2, 0.3], [0.75, 0.85], [1, 1.2]])`. It has vertices (*blue*) $(0.2, 0.85)$, $(0.3, 0.75)$, $(0.3, 0.85)$, $(0.2, 0.8)$, $(0.25, 0.75)$, whereas the other basic solutions (*red*) $(0.2, 0.75)$, $(0.2, 1)$, $(0.3, 0.9)$, $(0.35, 0.85)$, $(0.45, 0.75)$ are filtered out because they are infeasible. The face F has projections (*gray shadows*) $I' = p_1(F) = [0.2, 0.3]$ (*top border*), $J' = p_2(F) = [0.75, 0.85]$ (*left border*), and $K' = p_3(F) = [1, 1.15]$ (*right border*). Note that $K' \subsetneq K$.

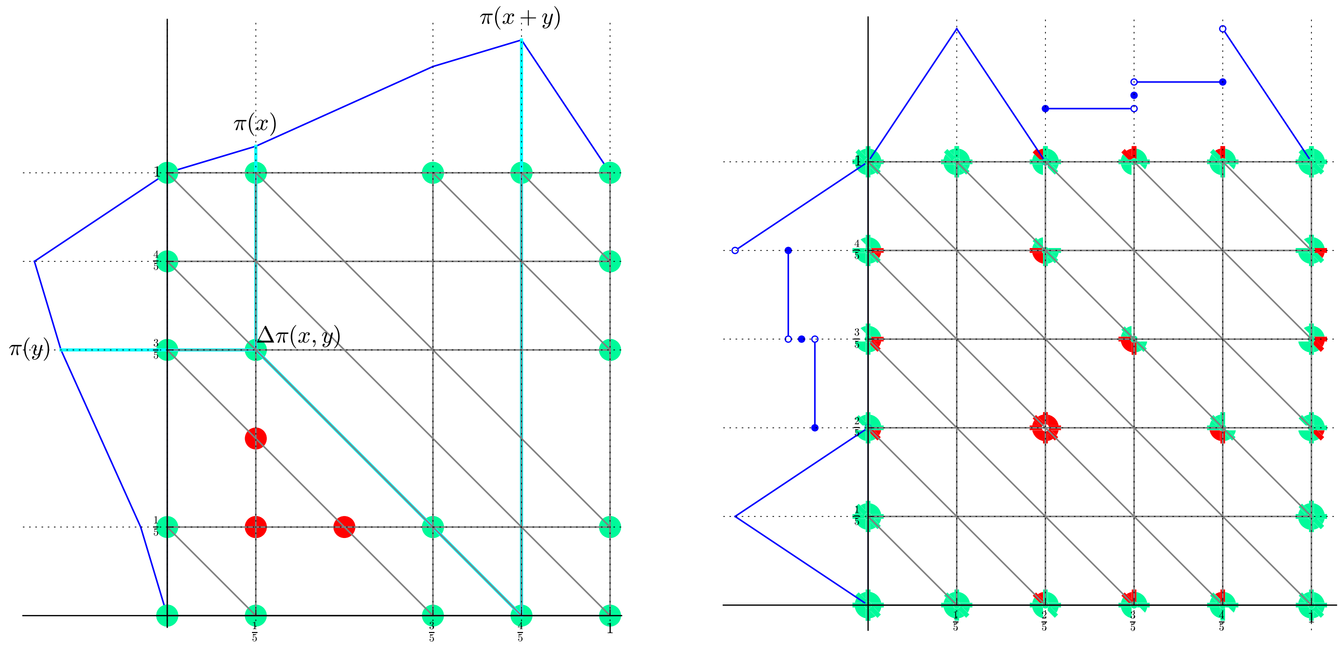


FIGURE 3. Two diagrams of functions and their polyhedral complexes $\Delta\mathcal{P}$ with colored cones at $\mathrm{vert}(\Delta\mathcal{P})$, as plotted by the command `plot_2d_diagram_with_cones(h)`. *Left*, continuous function `h = not_minimal_2()`. *Right*, random discontinuous function `h = equiv5_random_discont_1()`.

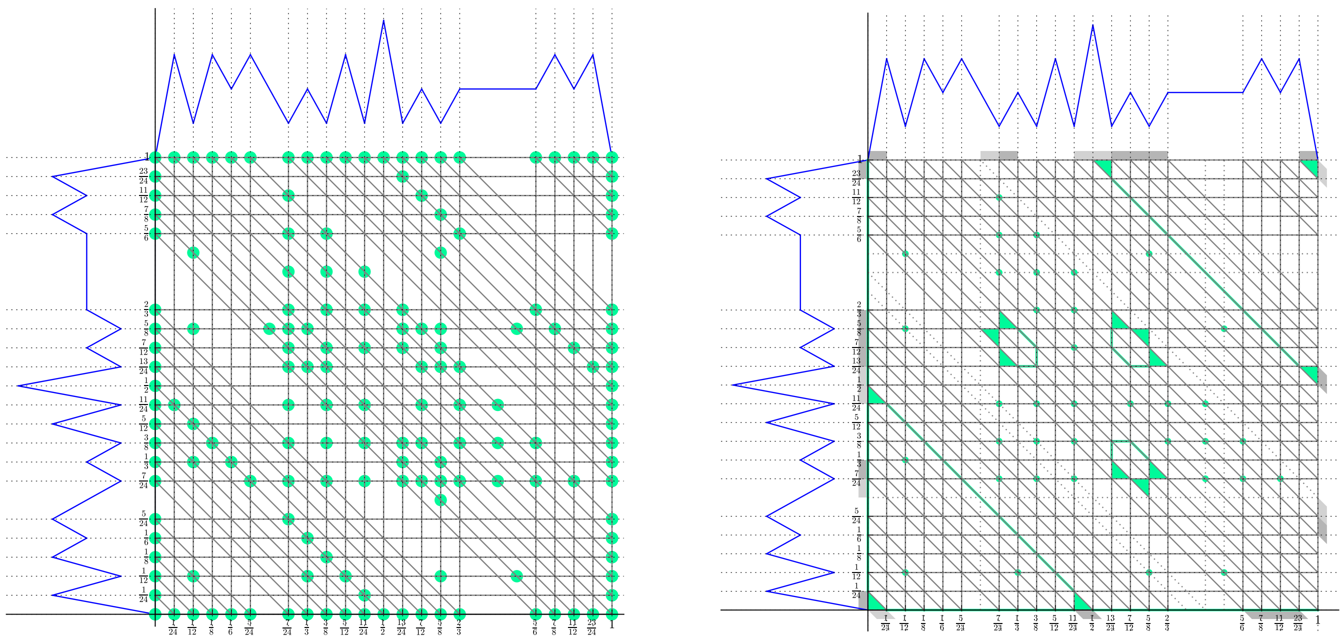


FIGURE 4. Diagrams of $\Delta\mathcal{P}$ of a continuous function `h = example7slopecoarse2()`, with (*left*) additive vertices as plotted by the command `plot_2d_diagram_with_cones(h)`; (*right*) maximal additive faces as plotted by the command `plot_2d_diagram(h)`.

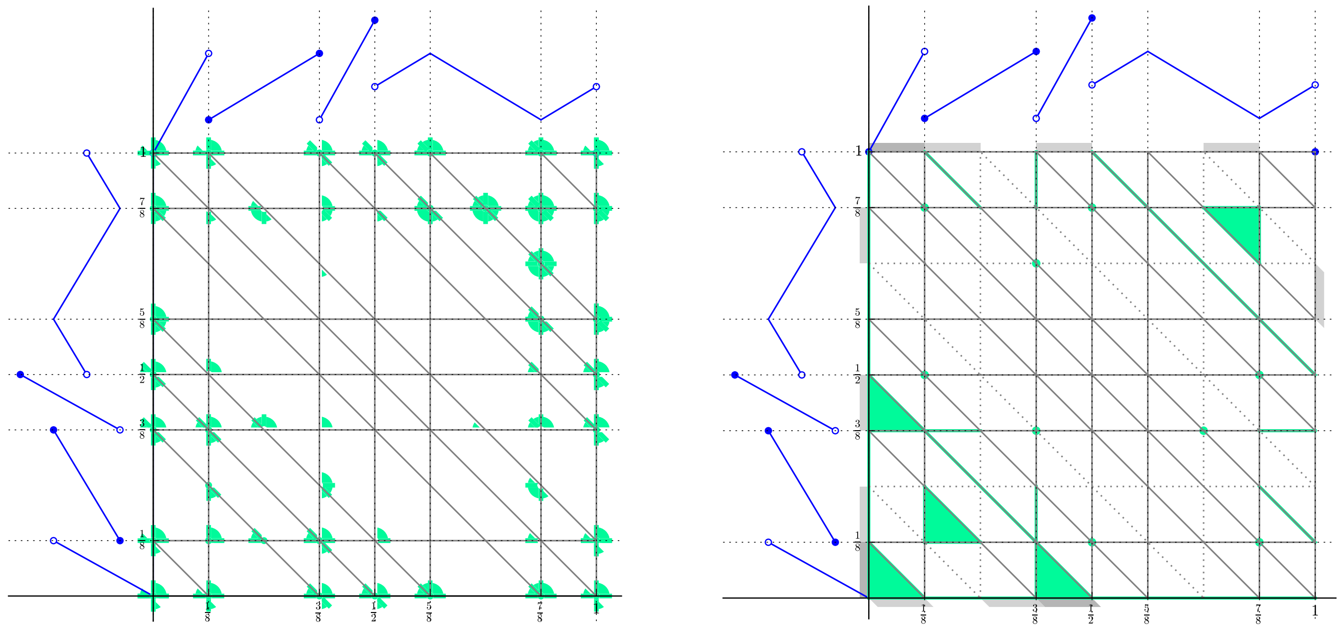


FIGURE 5. Diagrams of $\Delta\mathcal{P}$ of a discontinuous function `h = hildebrand_discont_3_slope_1()`, with (*left*) additive limiting cones as plotted by the command `plot_2d_diagram_with_cones(h)`; (*right*) additive faces as plotted by the command `plot_2d_diagram(h)`.

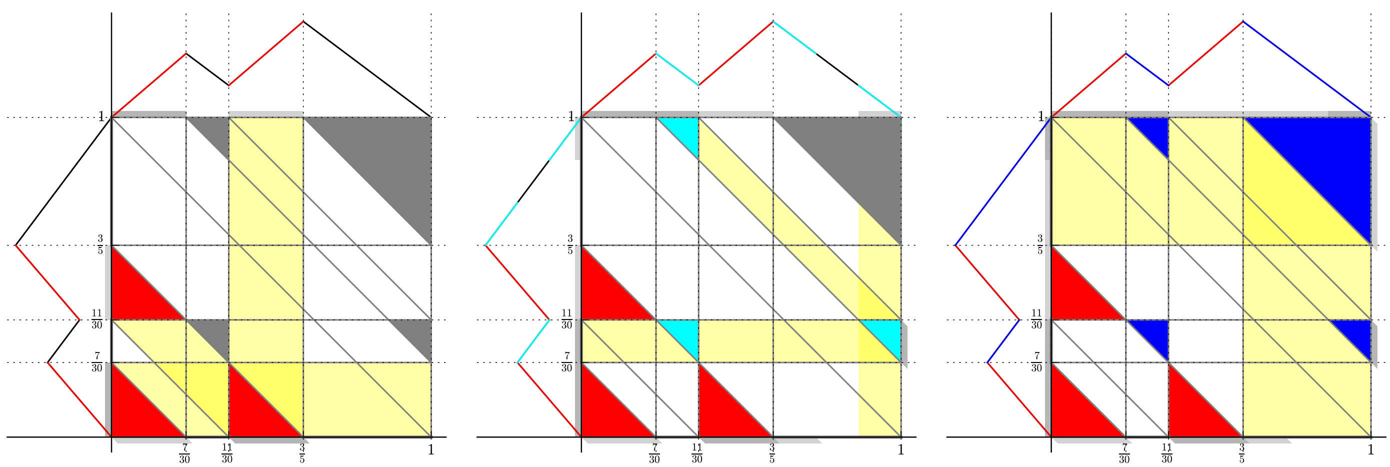


FIGURE 6. Compute the (directly) covered intervals for $\pi =$ `gj_2_slope(3/5,1/3)`.

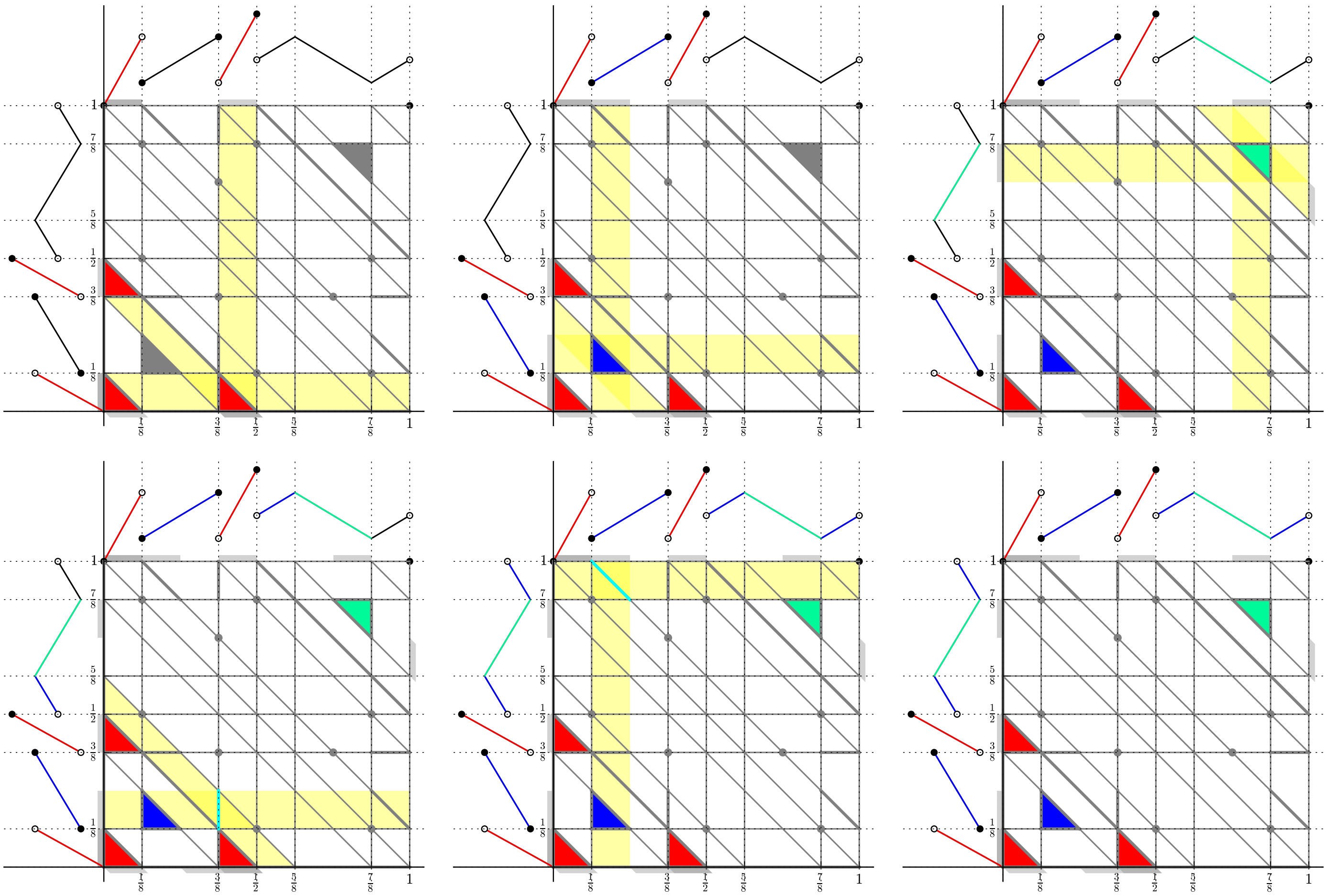


FIGURE 7. Compute the (directly and indirectly) covered intervals for $\pi =$ `hildebrand_discont_3_slope_1()`

TABLE 3. A sample Sage session on the extremality test

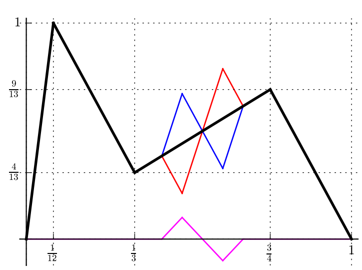
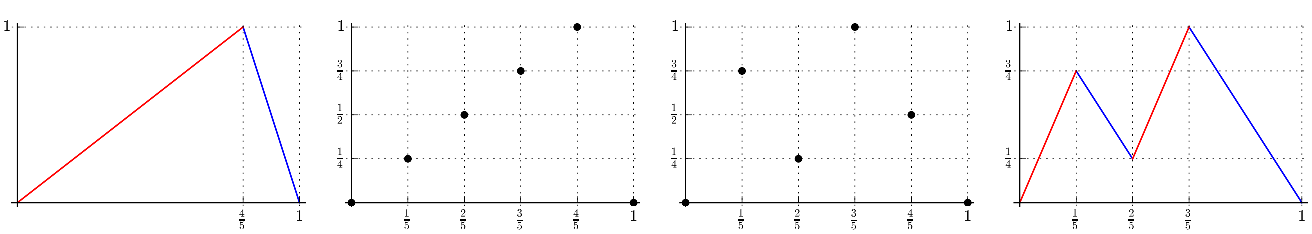


TABLE 4. A sample Sage session on discrete functions for the finite group problem.



6. Figures from *Equivariant Perturbation in Gomory and Johnson's Infinite Group Problem. VI. The Curious Case of Two-Sided Discontinuous Minimal Valid Functions*

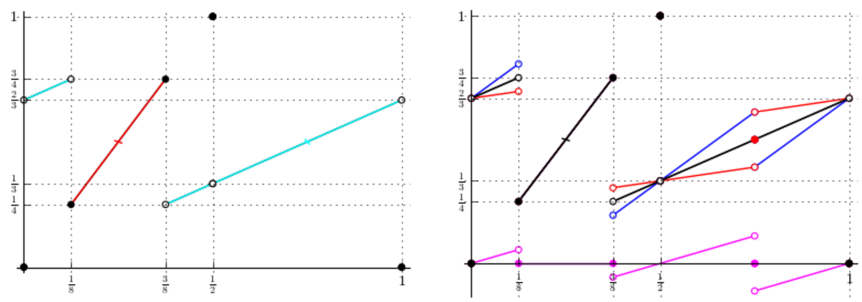


Figure 1. This function, π = zhou_two_sided_discontinuous_cannot_assume_any_continuity, is minimal, but not extreme, as proved by extremality_test(π, show_plots=True). The procedure first shows that for any distinct minimal $\pi^1 = \pi + \bar{\pi}$ (*blue*), $\pi^2 = \pi - \bar{\pi}$ (*red*) such that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$, the functions π^1 and π^2 are piecewise linear with the same breakpoints as π and possible additional breakpoints at $\frac{1}{4}$ and $\frac{3}{4}$. The open intervals between these breakpoints are covered. A finite-dimensional extremality test then finds exactly one linearly independent perturbation $\bar{\pi}$ (*magenta*), as shown. Thus all nontrivial perturbations are discontinuous at $\frac{3}{4}$, a point where π is continuous.

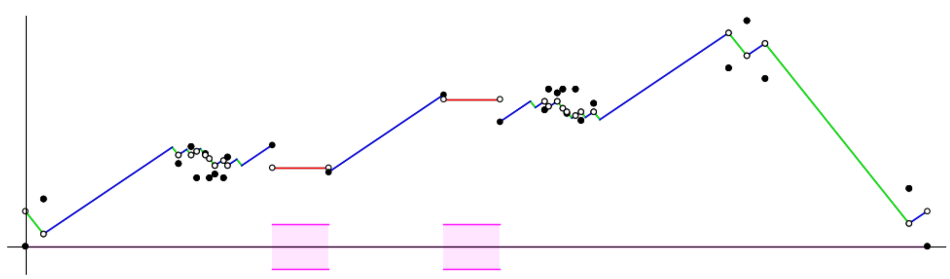


Figure 2. This function, π = kzh_minimal_has_only_crazy_perturbation_1, has three slopes (*blue*, *green*, *red*) and is discontinuous on both sides of the origin. It is a non-extreme minimal valid function, but in order to demonstrate non-extremality, one needs to use a highly discontinuous (locally microperiodic) perturbation. We construct a simple explicit example perturbation $\varepsilon\bar{\pi}$ (*magenta*). It takes three values, ε, 0, and $-\varepsilon$ (*horizontal magenta line segments*) where $\varepsilon = 0.0003$; in the figure it has been rescaled to amplitude $\frac{1}{10}$.

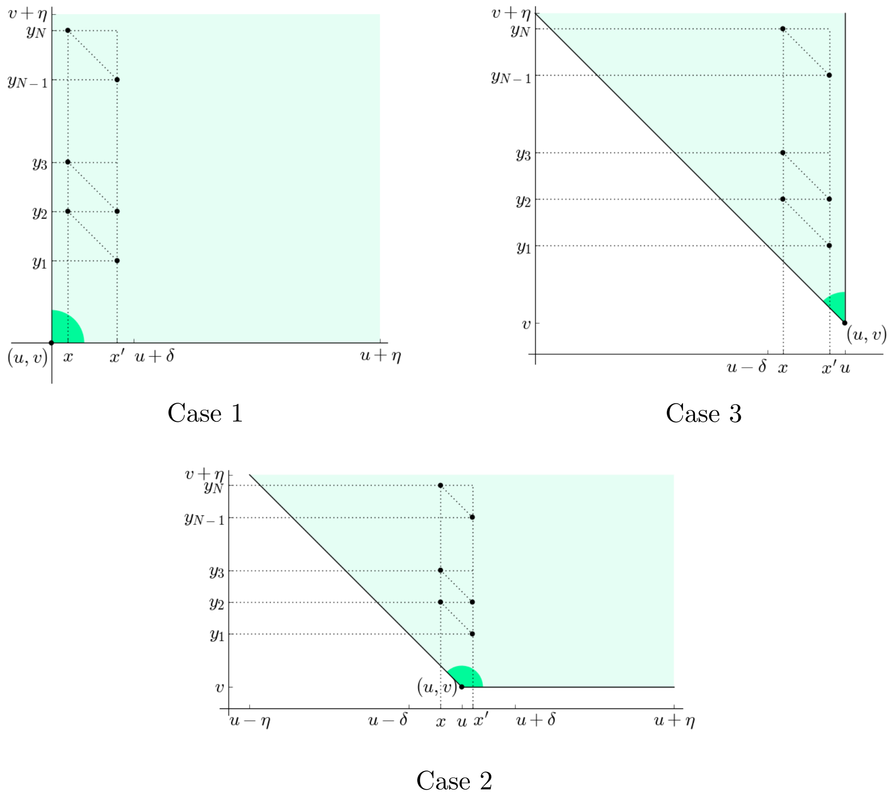


FIGURE 3. Illustration of the proof for U. Three partial diagrams of $\Delta\mathcal{P}$, where the tangent cone C of a two-dimensional face $F \in \Delta\mathcal{P}$ at vertex (u, v) is a (*left*, Case 1): right-angle cone (first quadrant); (*bottom*, Case 2): obtuse-angle cone; (*right*, Case 3): sharp-angle cone (contained in a second quadrant). The *light green area* C_η is contained in the face F. The *green sector* at (u, v) indicates that $\Delta\pi_F(u, v) = 0$. The *black points* inside the *light green area* show the sequences used in the proof.

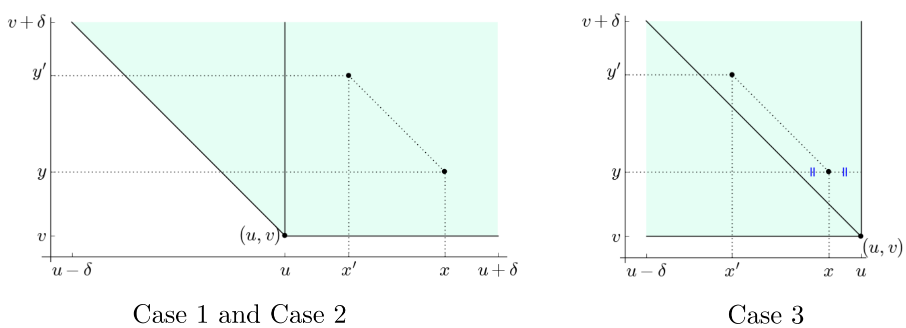
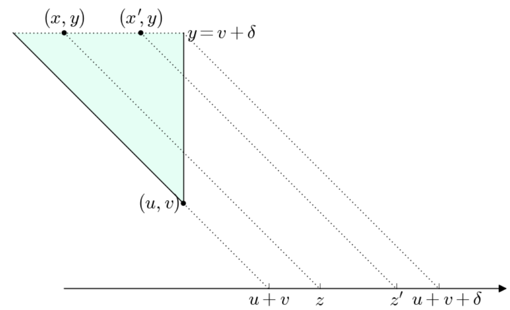
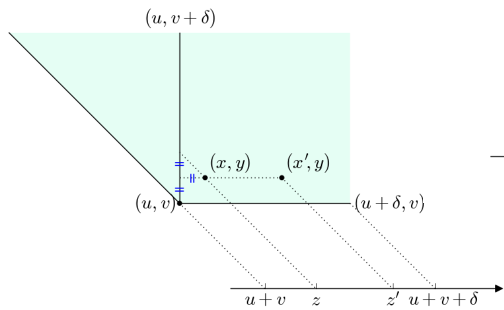


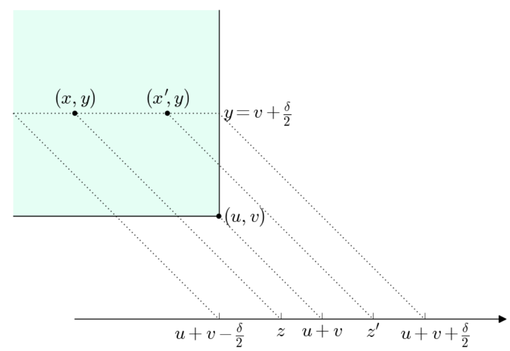
FIGURE 4. Illustration of the proof for V.



Case 3a



Case 1 and Case 2



Case 3b

Figure 5. Illustration of the proof for W