Homework 7

May 7, 2021

Contents

Problem 1.

(12 points) Solve the Dirichlet problem

$$\begin{aligned} -\Delta u &= 4, && \text{in } \Omega \\ u &= 0, && \text{on } \partial\Omega \end{aligned}$$

on the unit disk $\Omega = \{(x, y) \ : \ x^2 + y^2 \leq 1\}$ using quadratic isoparametric elements.

$$\sum_{j=1}^{n} \mathbf{u}_j \int_\Omega \nabla\phi_j \cdot \nabla\phi_i = \int_\Omega \phi_i f + \int_{\partial\Omega_N} \phi_i g_N - \sum_{j=n+1}^{n+n_\partial} \mathbf{u}_j \int_\Omega \nabla\phi_j \cdot \nabla\phi_i$$

Element stiffness matrix:

$$\begin{aligned} a_{ij}^{(k)} &= \int_{\Delta*} \left(b_2 \frac{\partial\psi_{*,i}}{\partial\xi} + b_3 \frac{\partial\psi_{*,i}}{\partial\eta} \right) \left(b_2 \frac{\partial\psi_{*,j}}{\partial\xi} + b_3 \frac{\partial\psi_{*,j}}{\partial\eta} \right) \frac{1}{|J_k|} \mathrm{d}\xi\mathrm{d}\eta \\ &+ \int_{\triangle*} \left(c_2 \frac{\partial\psi_{*,i}}{\partial\xi} + c_3 \frac{\partial\psi_{*,i}}{\partial\eta} \right) \left(c_2 \frac{\partial\psi_{*,j}}{\partial\xi} + c_3 \frac{\partial\psi_{*,j}}{\partial\eta} \right) \frac{1}{|J_k|} \mathrm{d}\xi\mathrm{d}\eta \end{aligned}$$

$$\begin{aligned} a_{ij}^{(k)} &= \int_{\Delta_k} \frac{\partial\psi_{k,i}}{\partial x} \frac{\partial\psi_{k,j}}{\partial x} + \frac{\partial\psi_{k,i}}{\partial y} \frac{\partial\psi_{k,j}}{\partial y} \ \mathrm{d}x \ \mathrm{d}y \quad i, j = 1, \ldots, n_k \\ &= \int_{\Delta*} \left\{ \frac{\partial\psi_{*,i}}{\partial x} \frac{\partial\psi_{*,j}}{\partial x} + \frac{\partial\psi_{*,i}}{\partial y} \frac{\partial\psi_{*,j}}{\partial y} \right\} |J_k| \, \mathrm{d}\xi\mathrm{d}\eta \end{aligned}$$

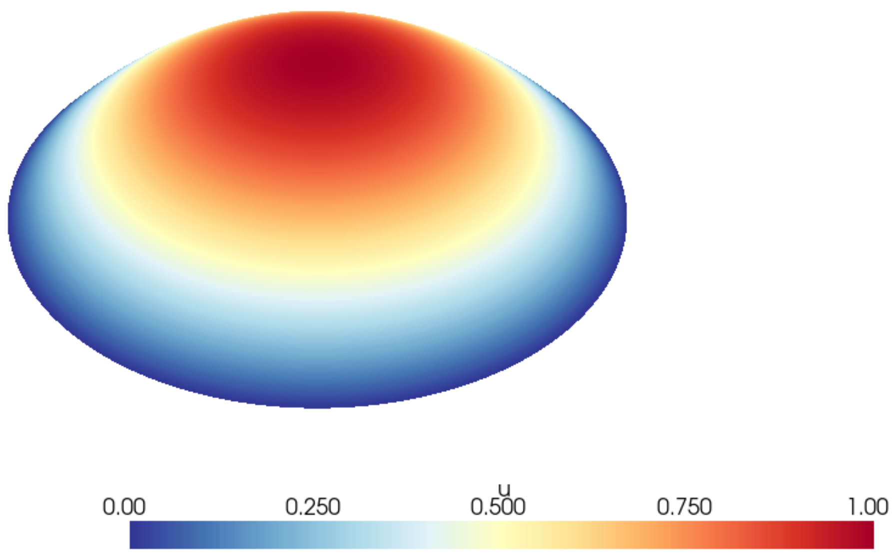


Figure 1: Finite element solution using order-2 Gaussian quadrature

What is the exact solution?

The exact solution is

$$\boxed{u = 1 - x^2 - y^2}$$

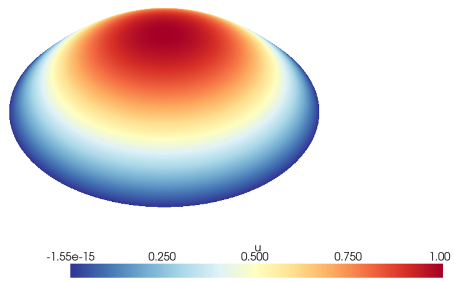
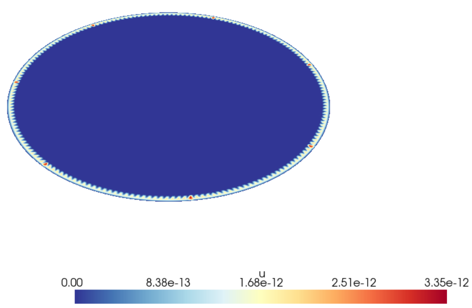
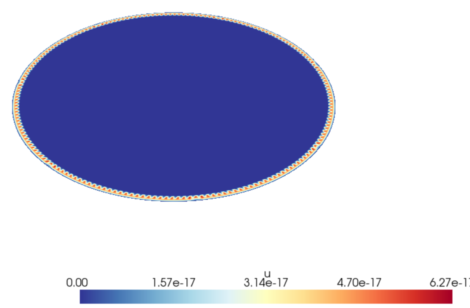


Figure 2: Closed-form solution to the stated Poisson problem.

Compute the H^1 seminorm and L^2 norm of the error for each of the meshes. Ignore the fringe error due to the piecewise parabolic mesh boundary not quite aligning with the circular domain.





Turn in a table of your errors and log-log plots of the error vs.~the mesh parameter, h.

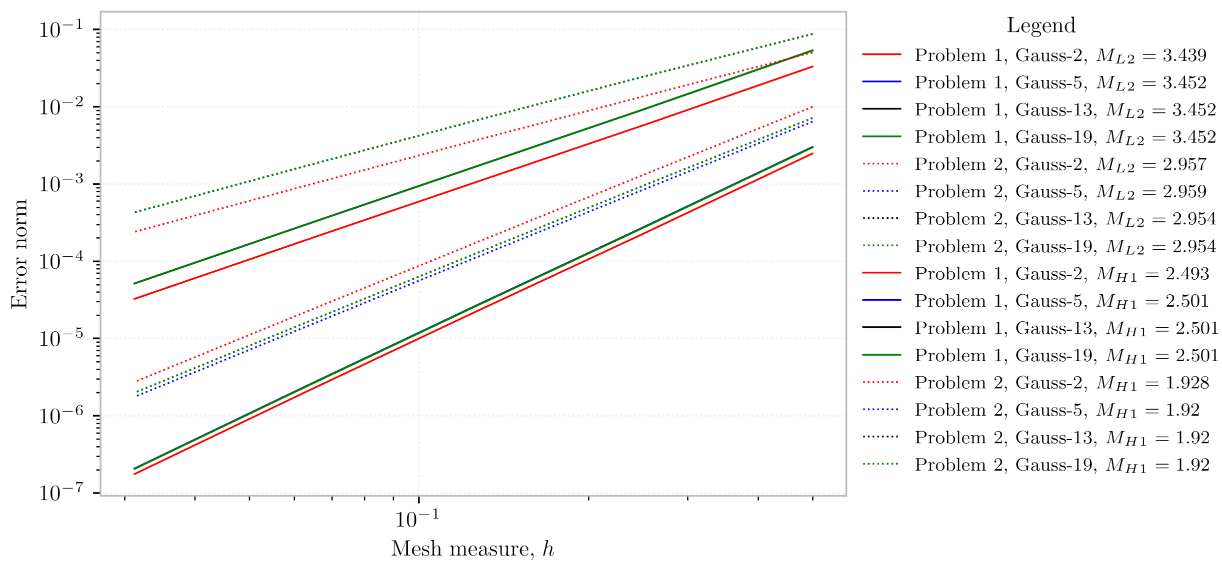


Table of errors in the L2 norm for problem 1

h	2	5	13	19
0.5	0.0024764578	0.0030141854	0.0029849953	0.0029849953
0.25	0.0002258162	0.00027375659	0.00027090969	0.00027090969
0.125	2.1527457e-05	2.5708536e-05	2.5448945e-05	2.5448945e-05
0.0625	1.9830323e-06	2.3460163e-06	2.3228069e-06	2.3228069e-06
0.03125	1.7587735e-07	2.0760524e-07	2.0555351e-07	2.0555351e-07

Table of errors in the L2 norm for problem 2

h	2	5	13	19
0.5	0.0099611923	0.0064076596	0.0071412137	0.0071412137
0.25	0.0013051996	0.00083984741	0.0009471105	0.0009471105
0.125	0.00016874334	0.00010830203	0.00012249798	0.00012249798
0.0625	2.1626624e-05	1.3882073e-05	1.5691434e-05	1.5691434e-05
0.03125	2.7380054e-06	1.756023e-06	1.9850051e-06	1.9850051e-06

Table of errors in the H1 norm for problem 1

h	2	5	13	19
0.5	0.033040555	0.05344312	0.053451002	0.053451002
0.25	0.0057268302	0.0091748492	0.0091751946	0.0091751946
0.125	0.0010299012	0.0016399388	0.0016399544	0.0016399544
0.0625	0.00018443725	0.00029237375	0.00029237444	0.00029237444

h	2	5	13	19
0.03125	3.254096e-05	5.152993e-05	5.1529961e-05	5.1529961e-05

Table of errors in the H1 norm for problem 2

h	2	5	13	19
0.5	0.050321736	0.087419539	0.087364004	0.087364004
0.25	0.013576873	0.024424342	0.024421502	0.024421502
0.125	0.0036081315	0.0065157661	0.006515551	0.006515551
0.0625	0.00093741811	0.0016853971	0.0016853811	0.0016853811
0.03125	0.00023986093	0.00042915864	0.00042915753	0.00042915753

Problem 2

Repeat (1) for the problem

$$\begin{aligned} -\Delta u &= \frac{\pi^2}{4}\left(\cos\frac{\pi r}{2} + \operatorname{sinc}\frac{\pi r}{2}\right), && \text{in } \Omega \\ u &= 0, && \text{on } \partial\Omega \end{aligned}$$

where Ω is again the unit disk. The exact solution is $u(x, y) = \cos\frac{\pi r}{2}$, where $r = \sqrt{x^2 + y^2}$.

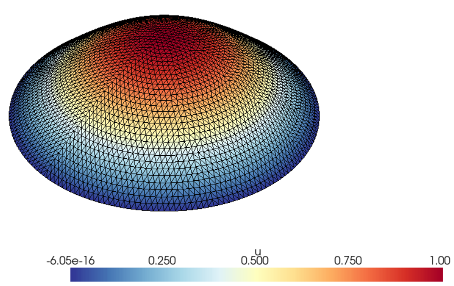


Figure 3: Plot of the given closed-form solution.

Mapped State Determination

$$A_{ij}$$

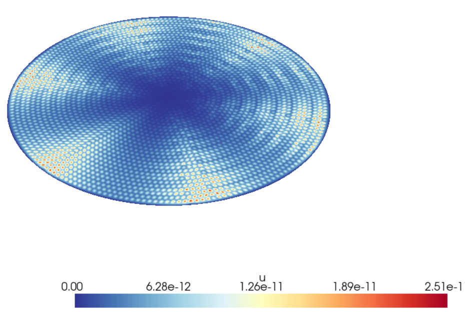
Problem 3

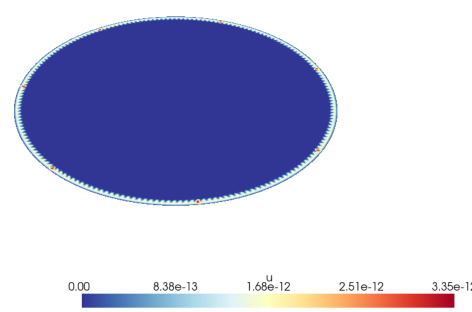
(6 points) Explain why the convergence rate for problem (1) is half an order higher than for problem (2). To understand what's going on, it may be helpful to plot the contribution to the H^1 and L^2 errors element by element. (see the sample file `plot_errors.m`). For example, in problem 2, I got the following values when I integrated

$$\iint_T \nabla(u_{FE} - u_{exact}) \cdot \nabla(u_{FE} - u_{exact})\, dx\, dy, \qquad (T \in \mathcal{T})$$

over the triangles in `circle_iso/mesh3`:

The H^1 semi-norm error in the finite element solution is the square root of the sum of all the errors shown here. (I got 0.00400512 for the H^1 error and 7.5355×10^{-5} for the L^2 error on this mesh).





Appendix

Solution plots

Plots of Finite Element Solutions

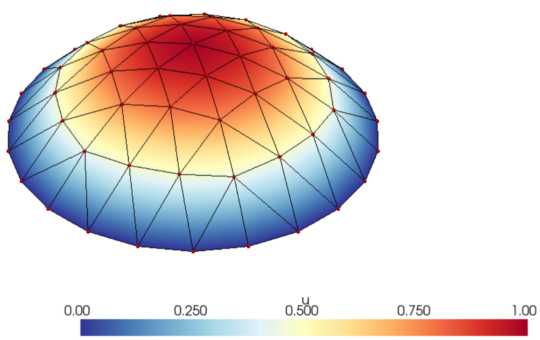


Figure 4: Finite element solution for problem 1 over mesh number 1 and order-2 numerical integration.

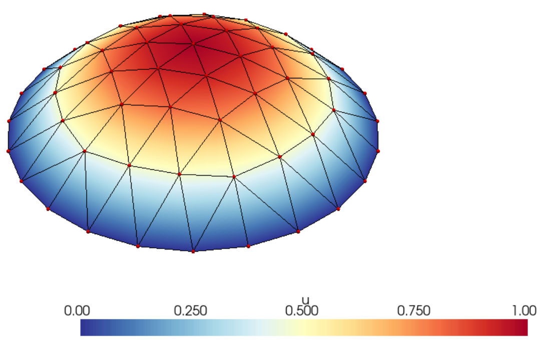


Figure 5: Finite element solution for problem 1 over mesh number 1 and order-5 numerical integration.

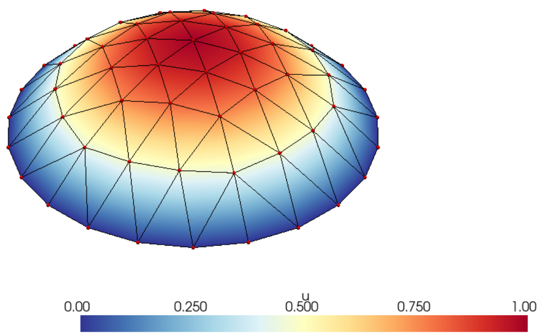


Figure 6: Finite element solution for problem 1 over mesh number 1 and order-8 numerical integration.

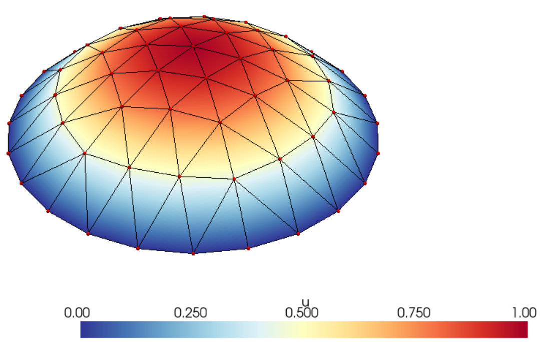


Figure 7: Finite element solution for problem 1 over mesh number 1 and order-13 numerical integration.

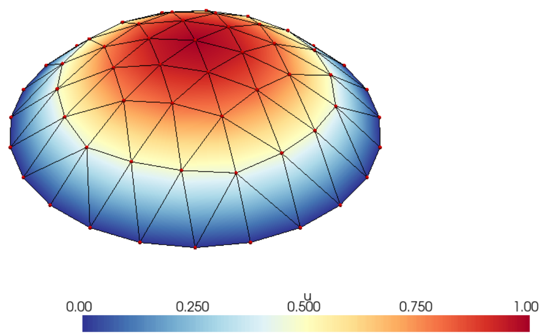


Figure 8: Finite element solution for problem 1 over mesh number 1 and order-19 numerical integration.

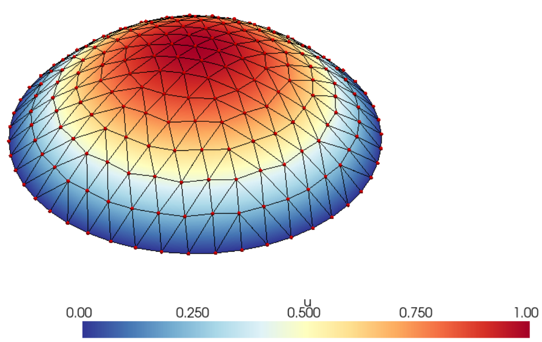


Figure 9: Finite element solution for problem 1 over mesh number 2 and order-2 numerical integration.

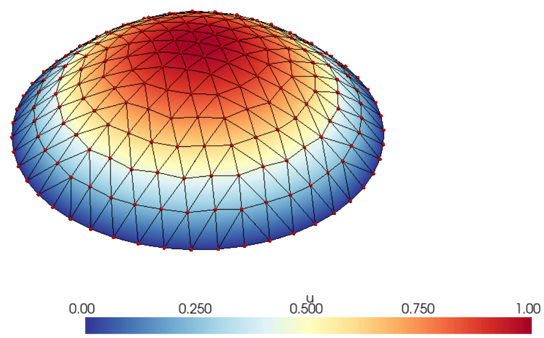


Figure 10: Finite element solution for problem 1 over mesh number 2 and order-5 numerical integration.

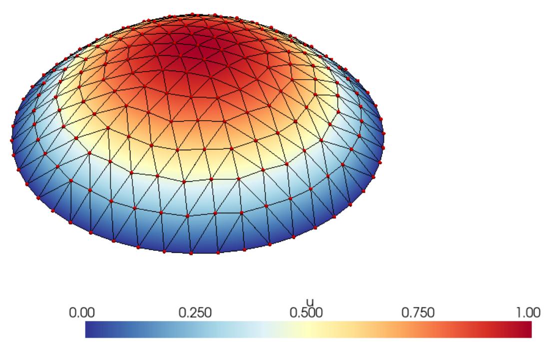


Figure 11: Finite element solution for problem 1 over mesh number 2 and order-8 numerical integration.

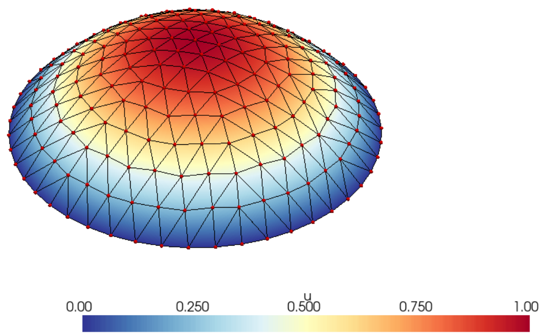


Figure 12: Finite element solution for problem 1 over mesh number 2 and order-13 numerical integration.

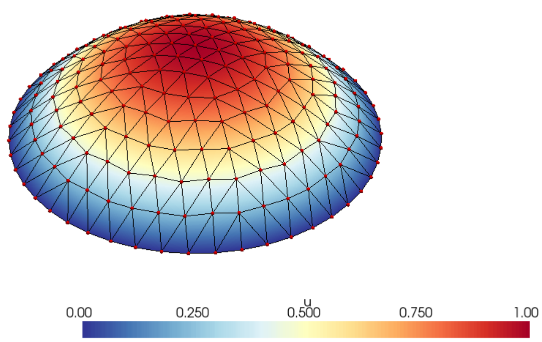


Figure 13: Finite element solution for problem 1 over mesh number 2 and order-19 numerical integration.

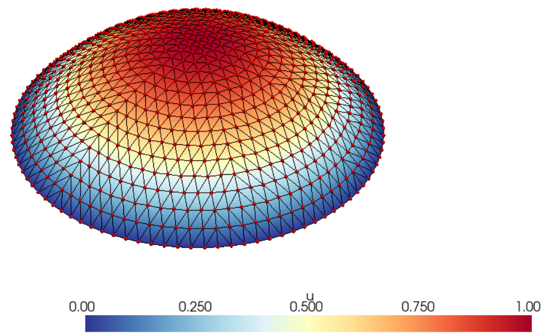


Figure 14: Finite element solution for problem 1 over mesh number 3 and order-2 numerical integration.

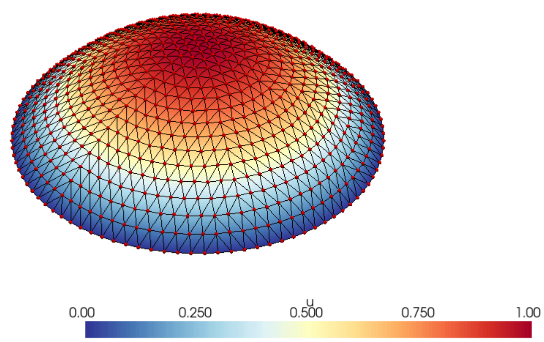


Figure 15: Finite element solution for problem 1 over mesh number 3 and order-5 numerical integration.

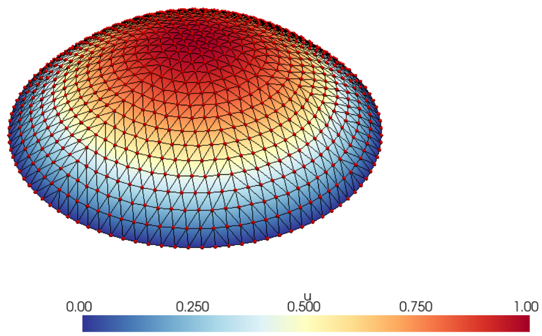


Figure 16: Finite element solution for problem 1 over mesh number 3 and order-8 numerical integration.

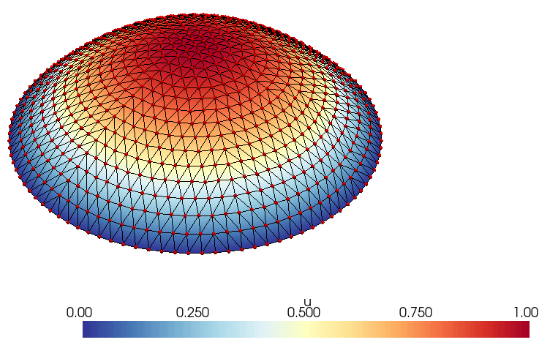


Figure 17: Finite element solution for problem 1 over mesh number 3 and order-13 numerical integration.

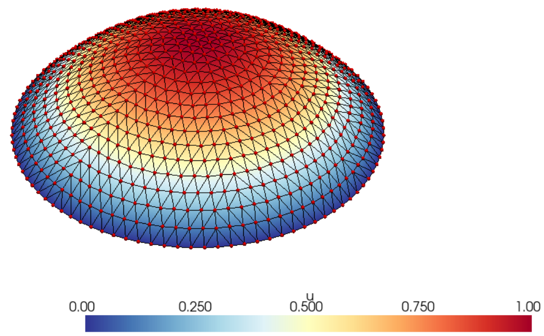


Figure 18: Finite element solution for problem 1 over mesh number 3 and order-19 numerical integration.

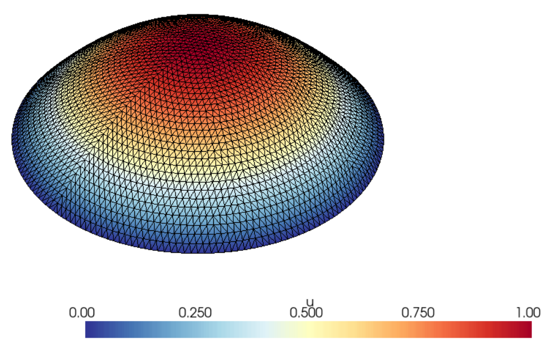


Figure 19: Finite element solution for problem 1 over mesh number 4 and order-2 numerical integration.

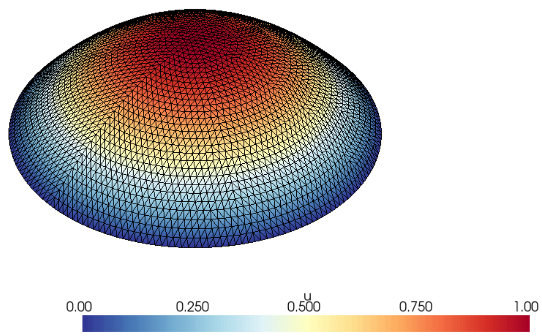


Figure 20: Finite element solution for problem 1 over mesh number 4 and order-5 numerical integration.

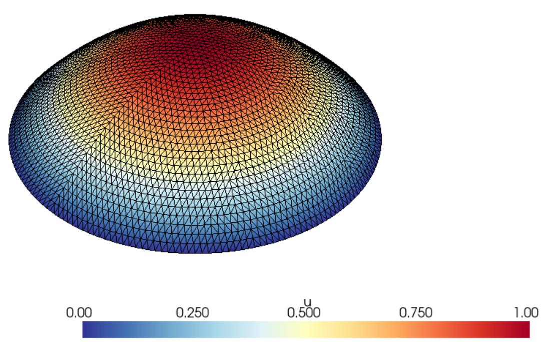


Figure 21: Finite element solution for problem 1 over mesh number 4 and order-8 numerical integration.

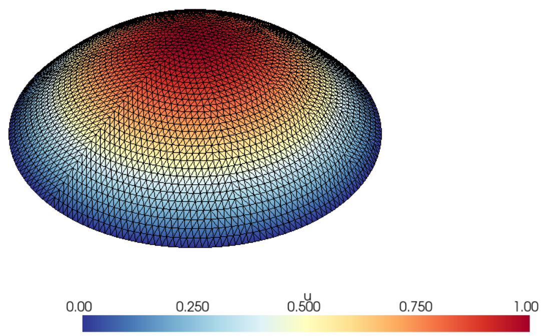


Figure 22: Finite element solution for problem 1 over mesh number 4 and order-13 numerical integration.

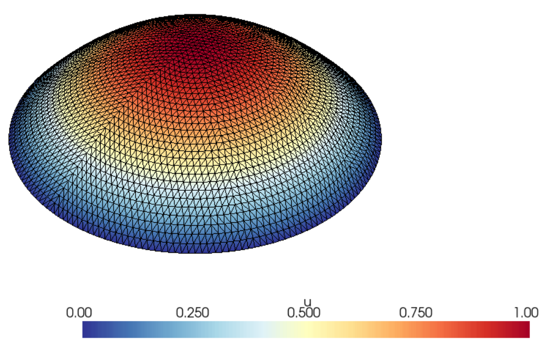


Figure 23: Finite element solution for problem 1 over mesh number 4 and order-19 numerical integration.

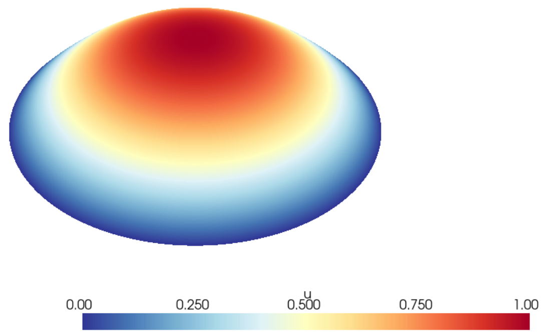


Figure 24: Finite element solution for problem 1 over mesh number 5 and order-2 numerical integration.

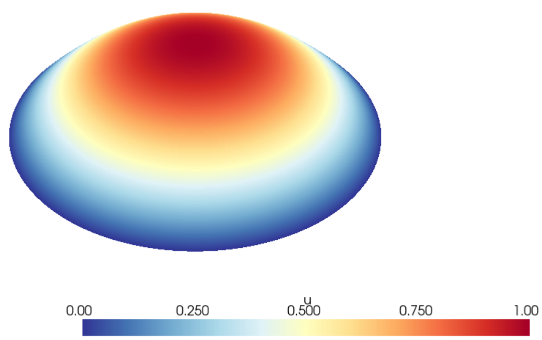


Figure 25: Finite element solution for problem 1 over mesh number 5 and order-5 numerical integration.

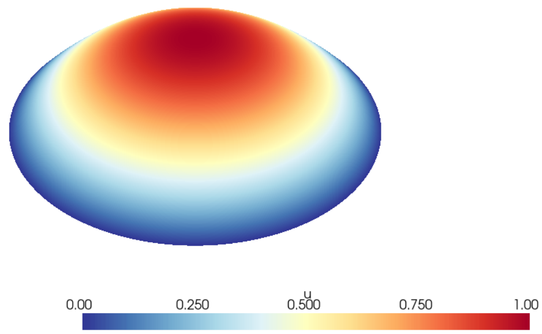


Figure 26: Finite element solution for problem 1 over mesh number 5 and order-8 numerical integration.

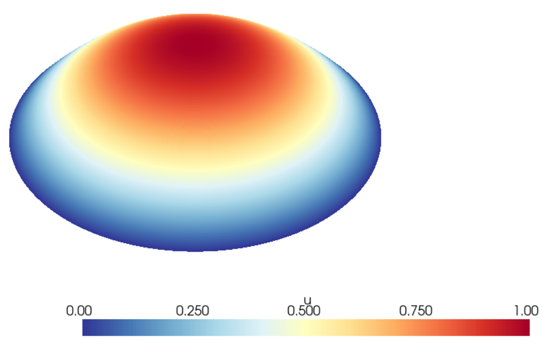


Figure 27: Finite element solution for problem 1 over mesh number 5 and order-13 numerical integration.

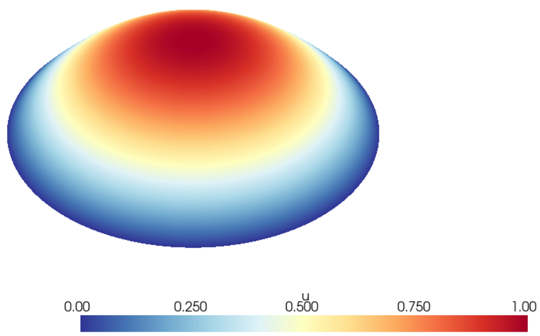


Figure 28: Finite element solution for problem 1 over mesh number 5 and order-19 numerical integration.

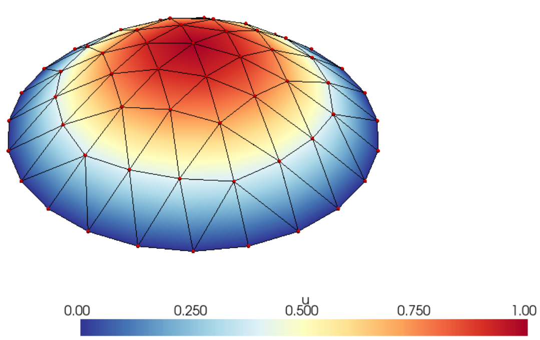


Figure 29: Finite element solution for problem 1 over mesh number 1 and order-2 numerical integration.

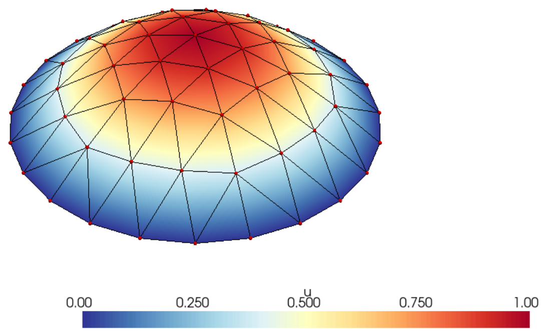


Figure 30: Finite element solution for problem 1 over mesh number 1 and order-5 numerical integration.

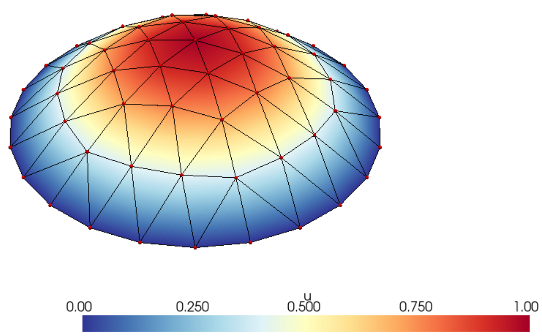


Figure 31: Finite element solution for problem 1 over mesh number 1 and order-8 numerical integration.

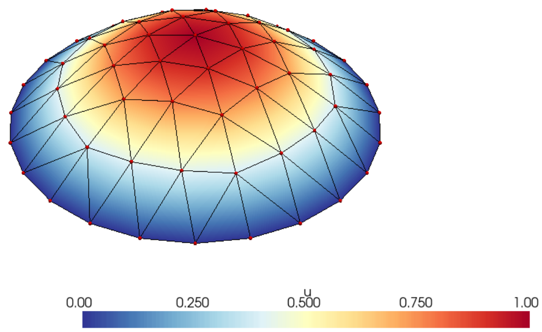


Figure 32: Finite element solution for problem 1 over mesh number 1 and order-13 numerical integration.

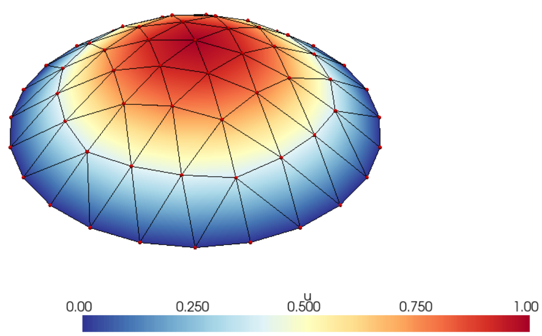


Figure 33: Finite element solution for problem 1 over mesh number 1 and order-19 numerical integration.

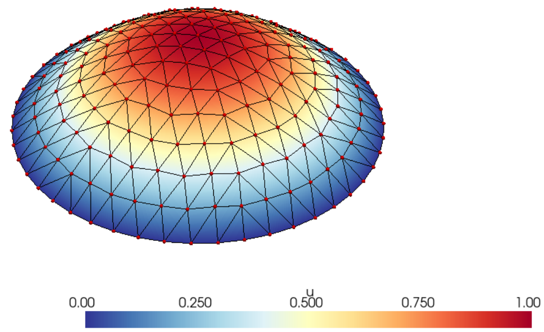


Figure 34: Finite element solution for problem 1 over mesh number 2 and order-2 numerical integration.

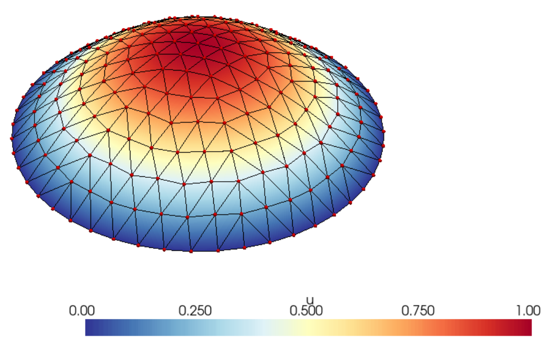


Figure 35: Finite element solution for problem 1 over mesh number 2 and order-5 numerical integration.

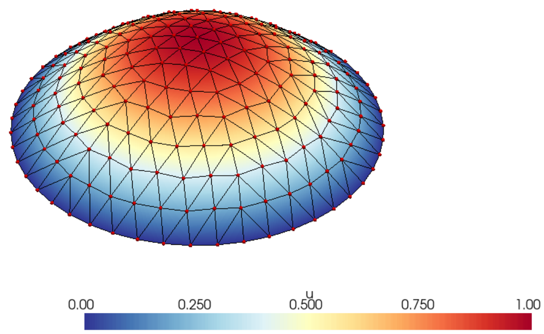


Figure 36: Finite element solution for problem 1 over mesh number 2 and order-8 numerical integration.

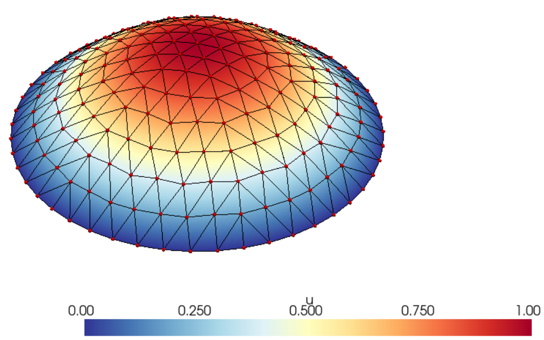


Figure 37: Finite element solution for problem 1 over mesh number 2 and order-13 numerical integration.

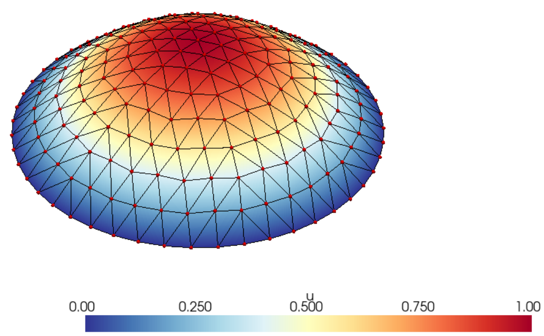


Figure 38: Finite element solution for problem 1 over mesh number 2 and order-19 numerical integration.

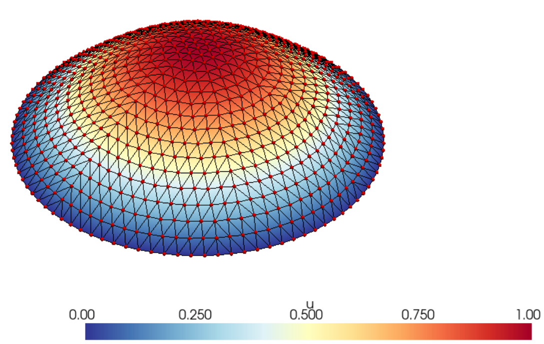


Figure 39: Finite element solution for problem 1 over mesh number 3 and order-2 numerical integration.

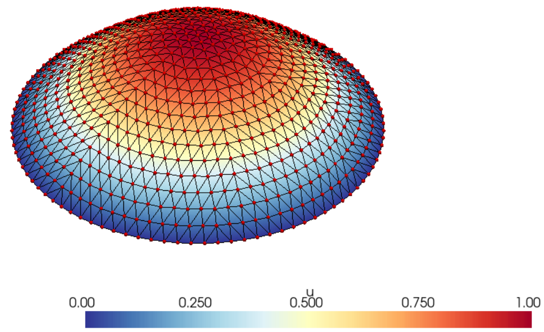


Figure 40: Finite element solution for problem 1 over mesh number 3 and order-5 numerical integration.

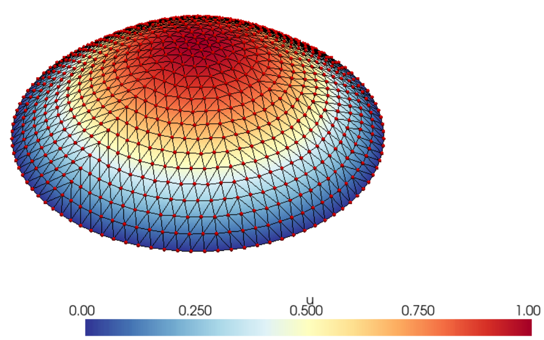


Figure 41: Finite element solution for problem 1 over mesh number 3 and order-8 numerical integration.

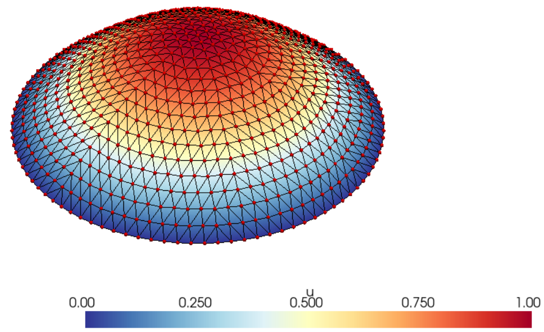


Figure 42: Finite element solution for problem 1 over mesh number 3 and order-13 numerical integration.

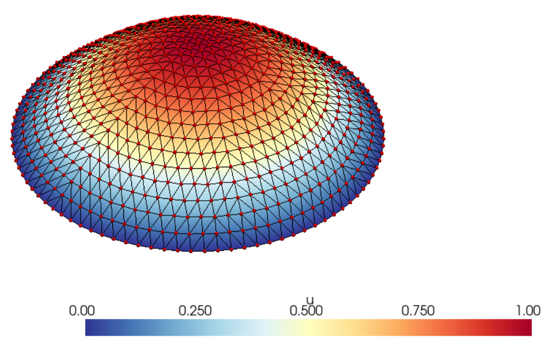


Figure 43: Finite element solution for problem 1 over mesh number 3 and order-19 numerical integration.

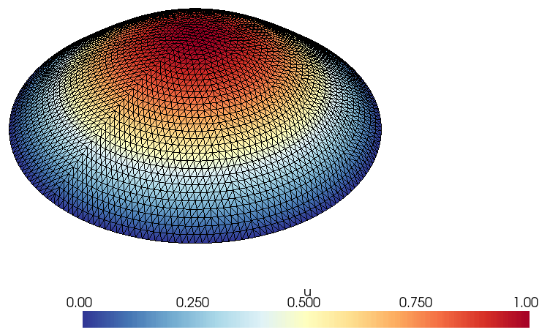


Figure 44: Finite element solution for problem 1 over mesh number 4 and order-2 numerical integration.

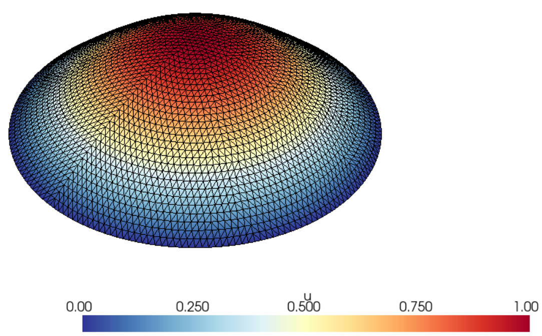


Figure 45: Finite element solution for problem 1 over mesh number 4 and order-5 numerical integration.

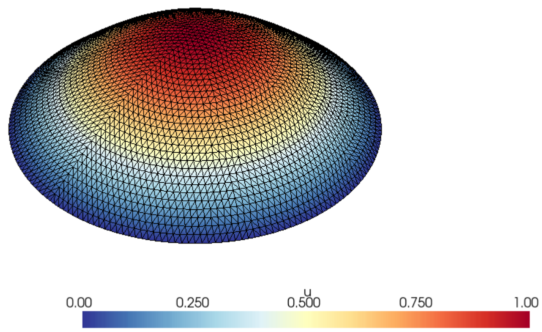


Figure 46: Finite element solution for problem 1 over mesh number 4 and order-8 numerical integration.

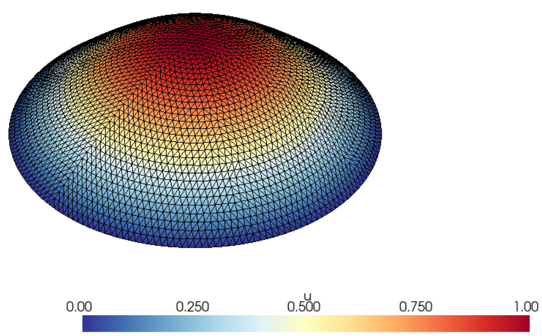


Figure 47: Finite element solution for problem 1 over mesh number 4 and order-13 numerical integration.

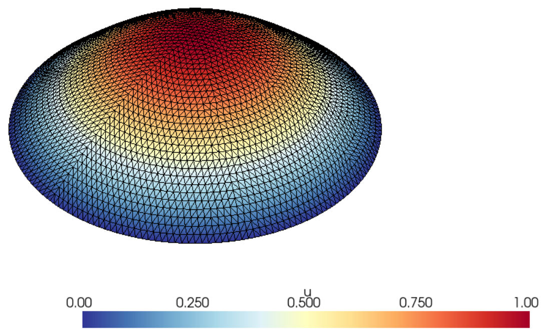


Figure 48: Finite element solution for problem 1 over mesh number 4 and order-19 numerical integration.

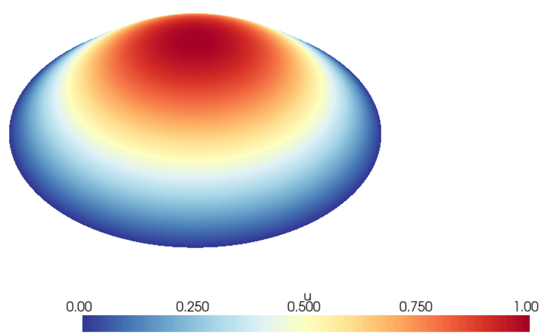


Figure 49: Finite element solution for problem 1 over mesh number 5 and order-2 numerical integration.

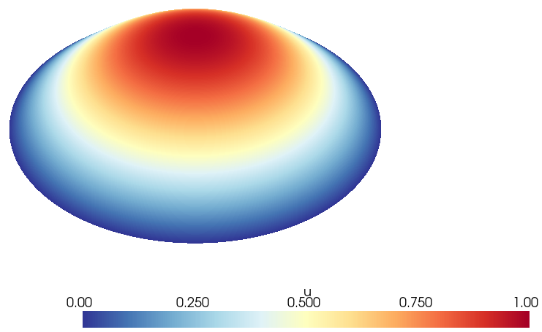


Figure 50: Finite element solution for problem 1 over mesh number 5 and order-5 numerical integration.

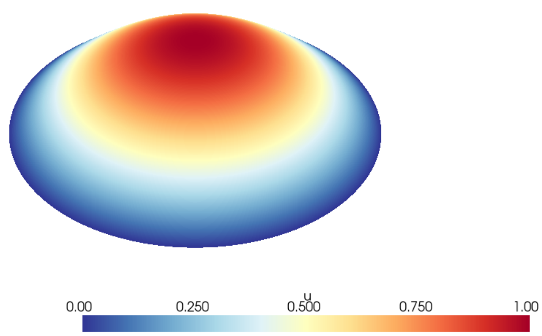


Figure 51: Finite element solution for problem 1 over mesh number 5 and order-8 numerical integration.

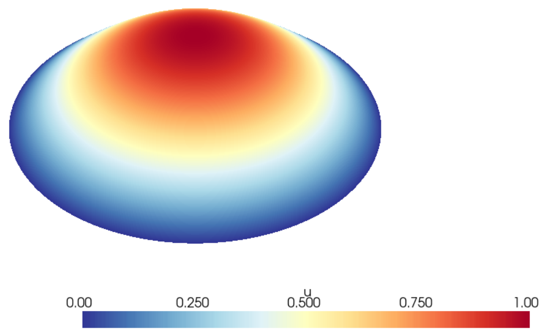


Figure 52: Finite element solution for problem 1 over mesh number 5 and order-13 numerical integration.

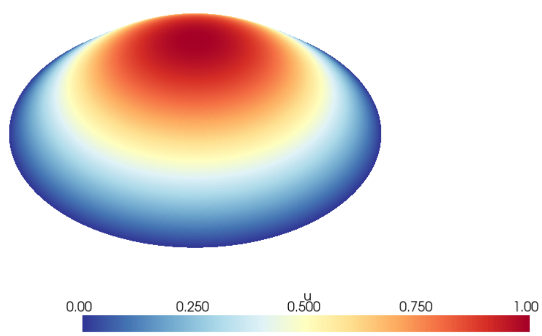


Figure 53: Finite element solution for problem 1 over mesh number 5 and order-19 numerical integration.

Errors in the H^1 and $L2$ norms

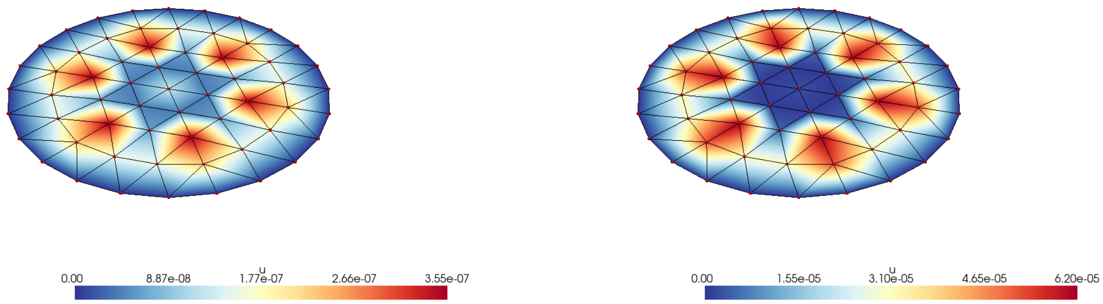


Figure 54: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 1 using order 2 quadrature.

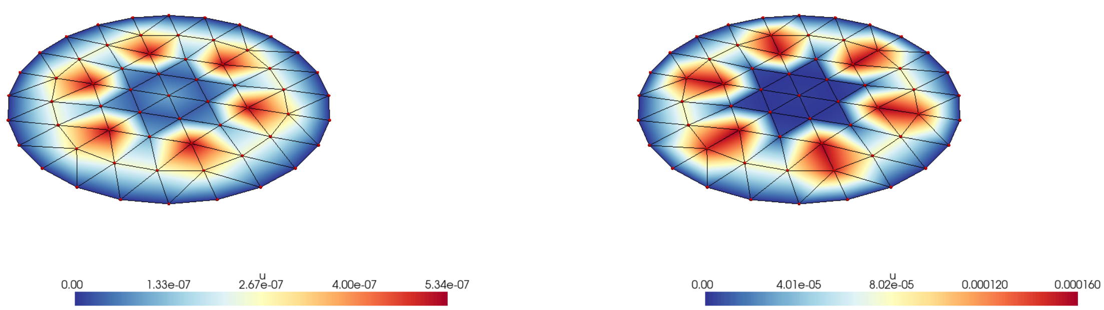


Figure 55: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 1 using order 5 quadrature.

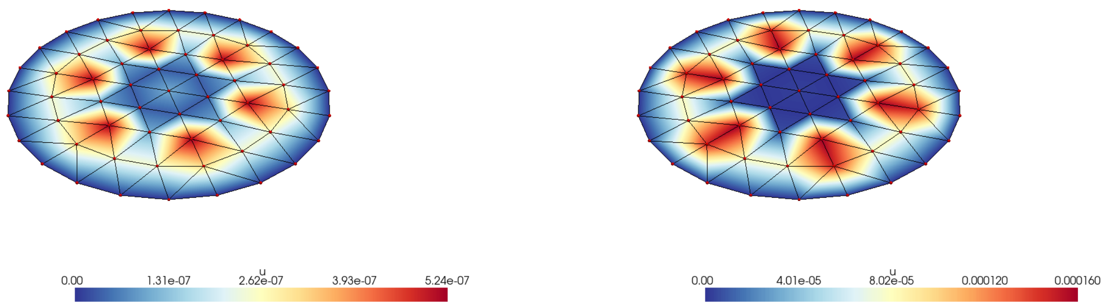


Figure 56: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 1 using order 8 quadrature.

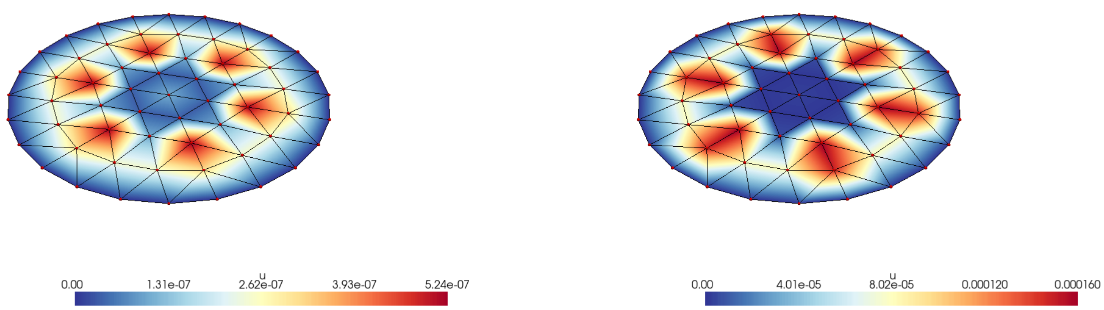


Figure 57: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 1 using order 13 quadrature.

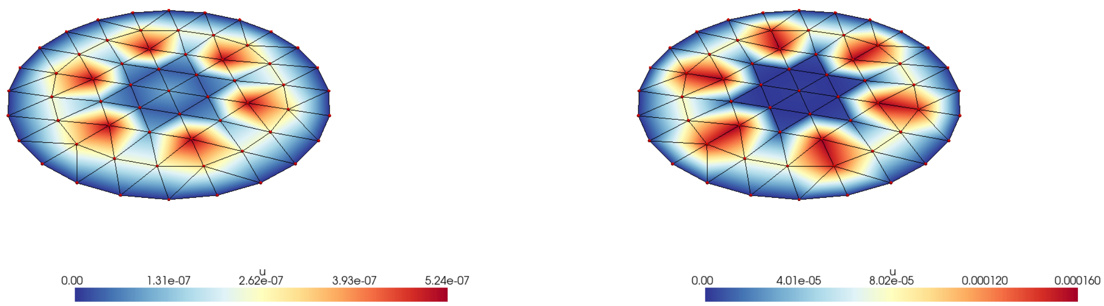


Figure 58: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 1 using order 19 quadrature.

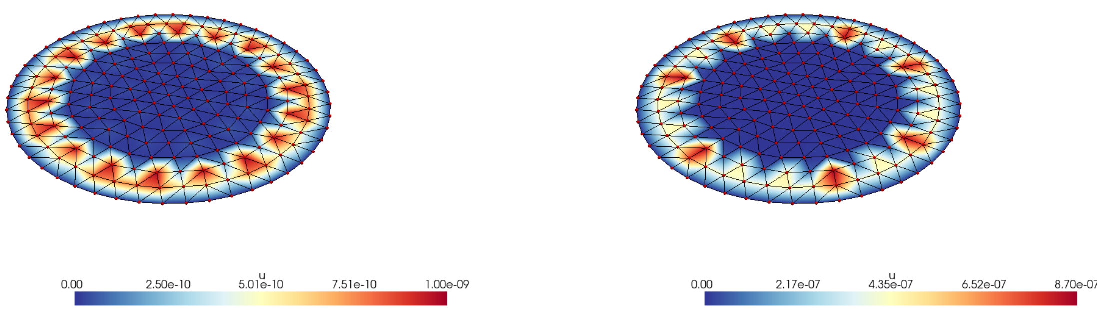


Figure 59: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 2 using order 2 quadrature.

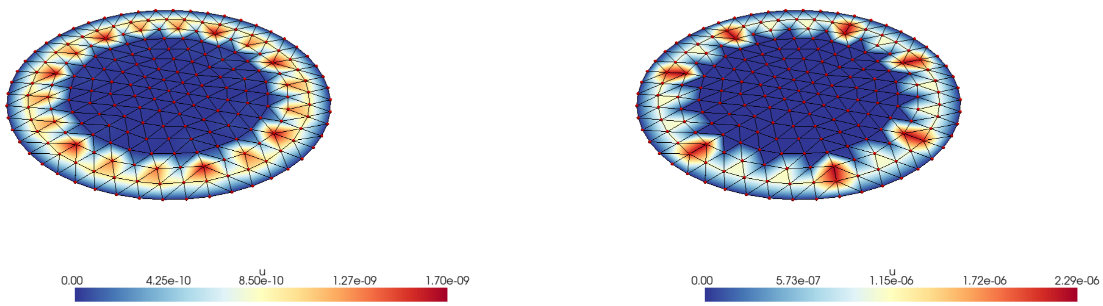


Figure 60: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 2 using order 5 quadrature.

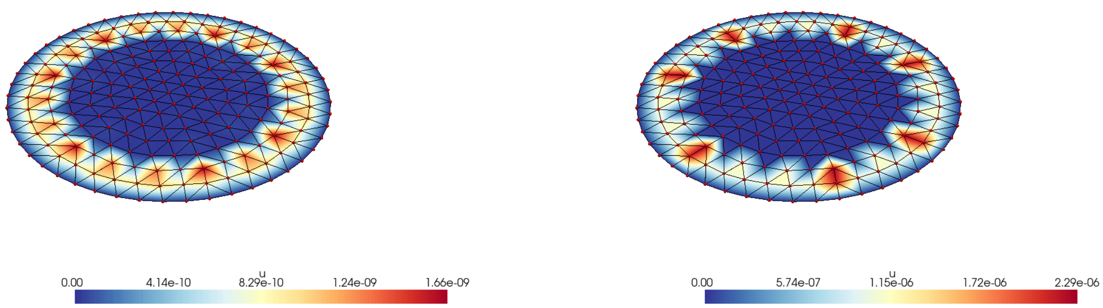


Figure 61: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 2 using order 8 quadrature.

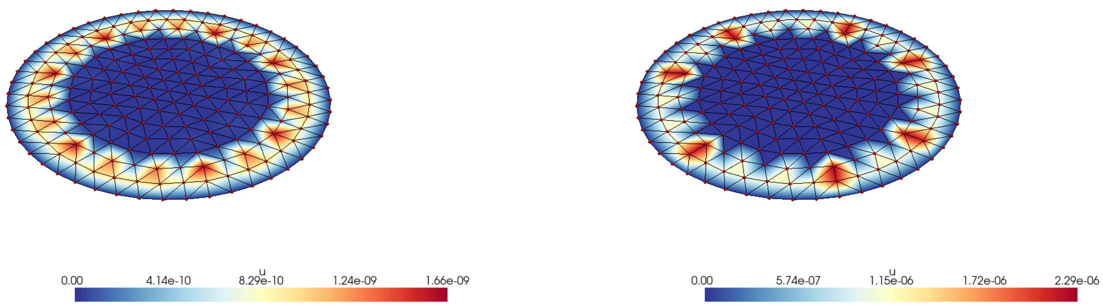


Figure 62: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 2 using order 13 quadrature.

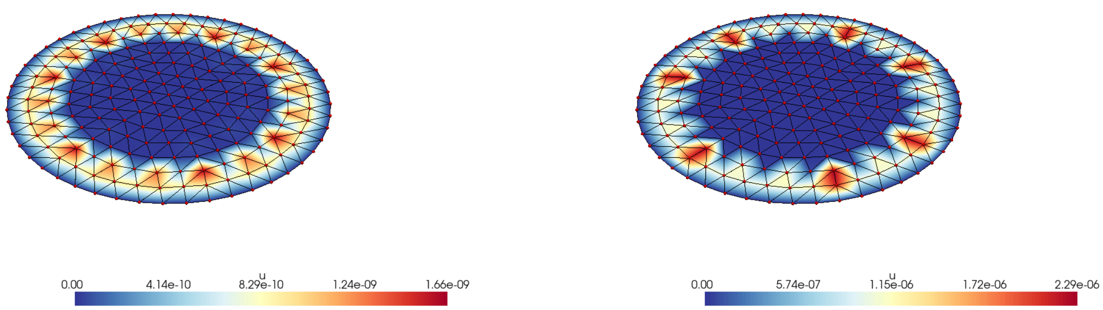


Figure 63: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 2 using order 19 quadrature.

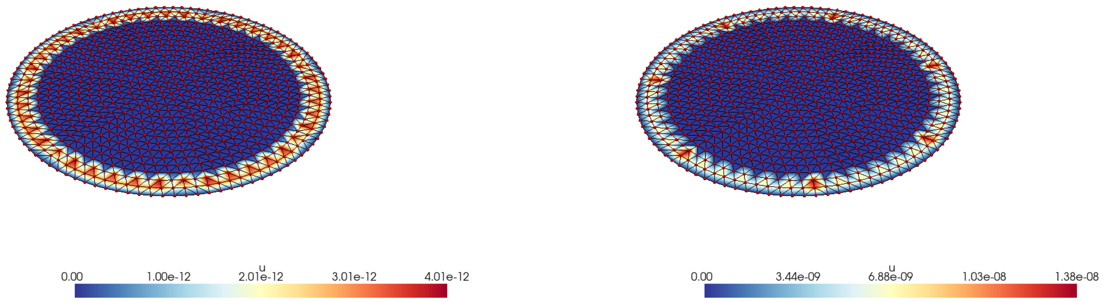


Figure 64: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 3 using order 2 quadrature.

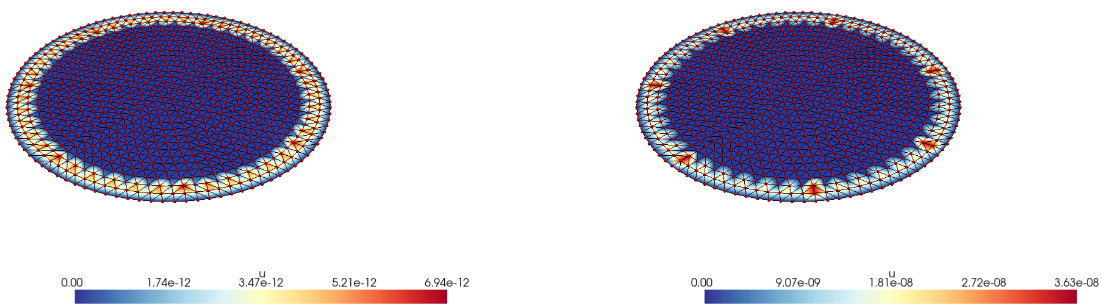


Figure 65: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 3 using order 5 quadrature.

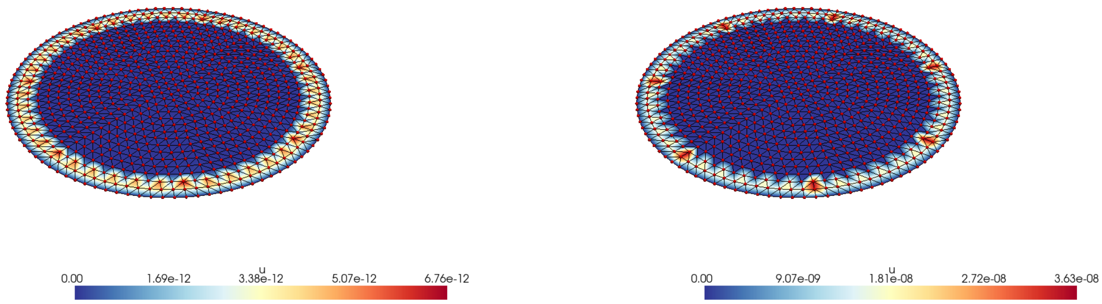


Figure 66: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 3 using order 8 quadrature.

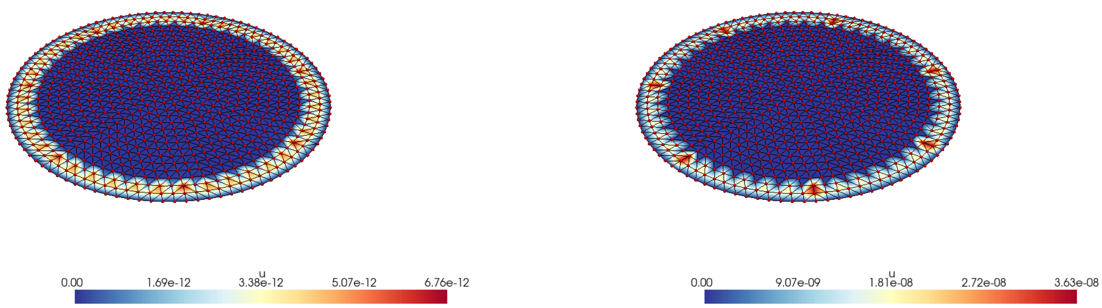


Figure 67: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 3 using order 13 quadrature.

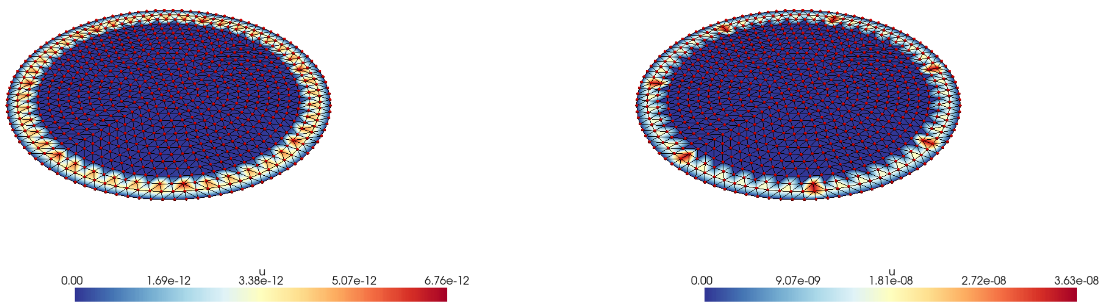


Figure 68: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 3 using order 19 quadrature.

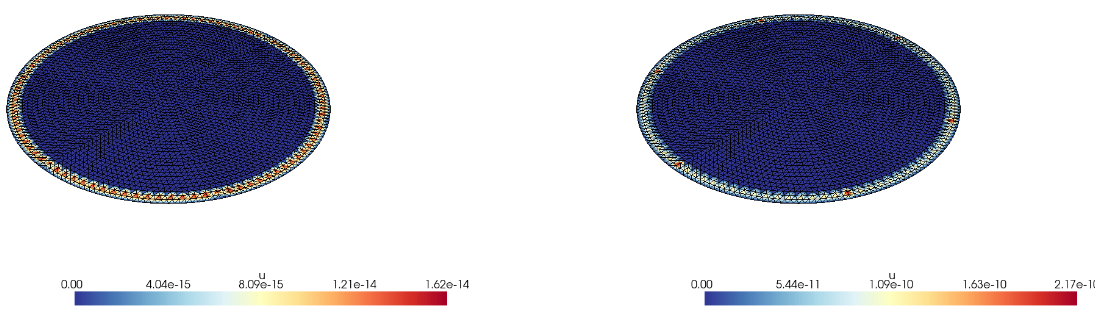


Figure 69: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 4 using order 2 quadrature.

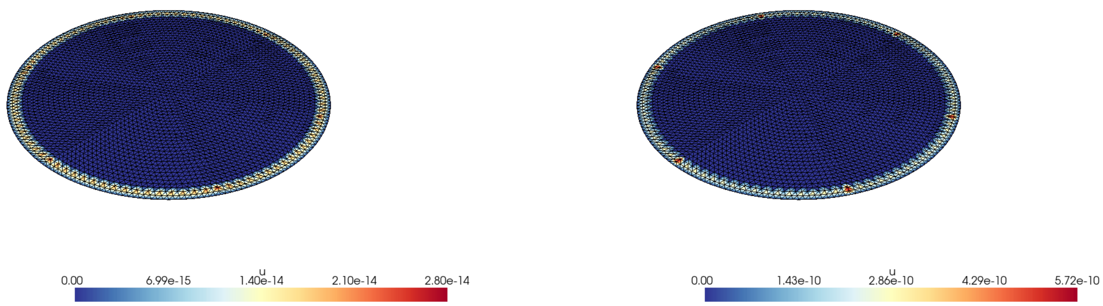


Figure 70: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 4 using order 5 quadrature.

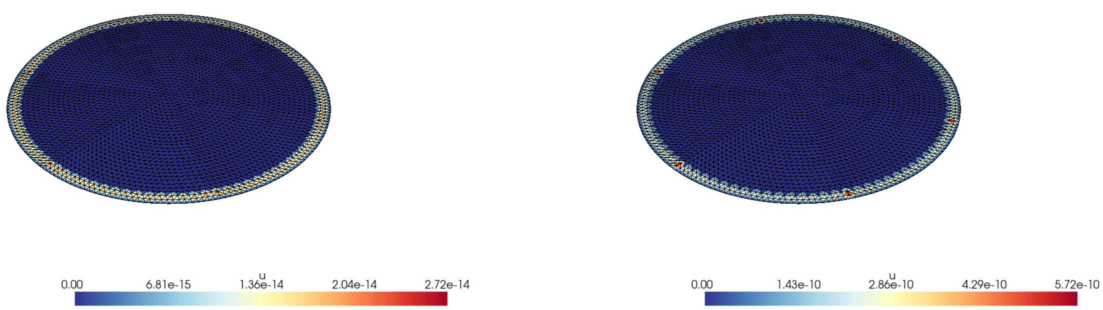


Figure 71: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 4 using order 8 quadrature.

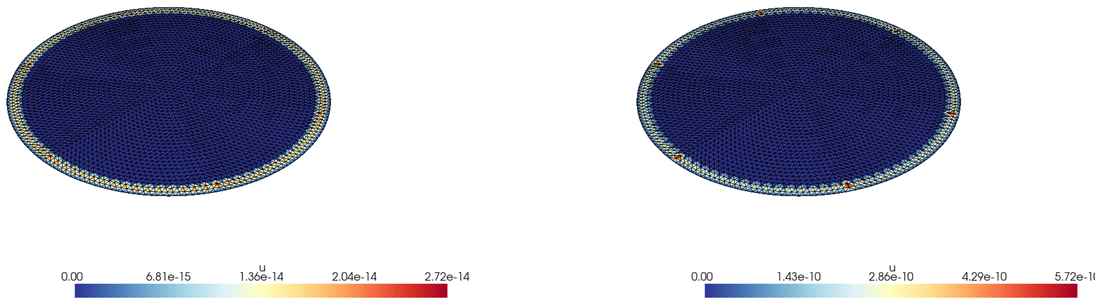


Figure 72: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 4 using order 13 quadrature.

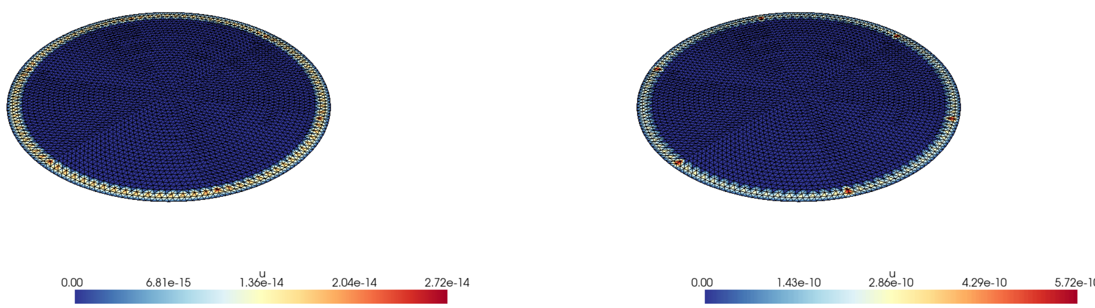


Figure 73: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 4 using order 19 quadrature.

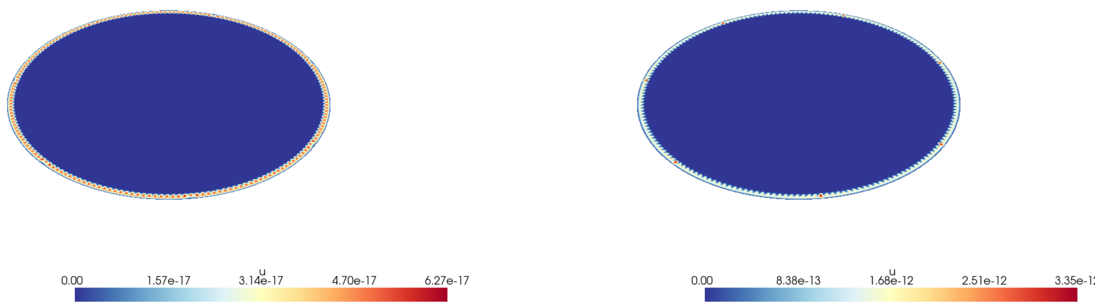


Figure 74: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 5 using order 2 quadrature.

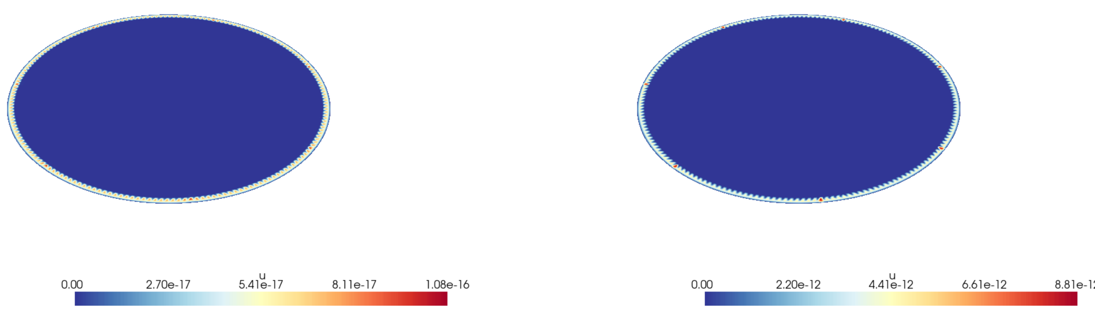


Figure 75: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 5 using order 5 quadrature.

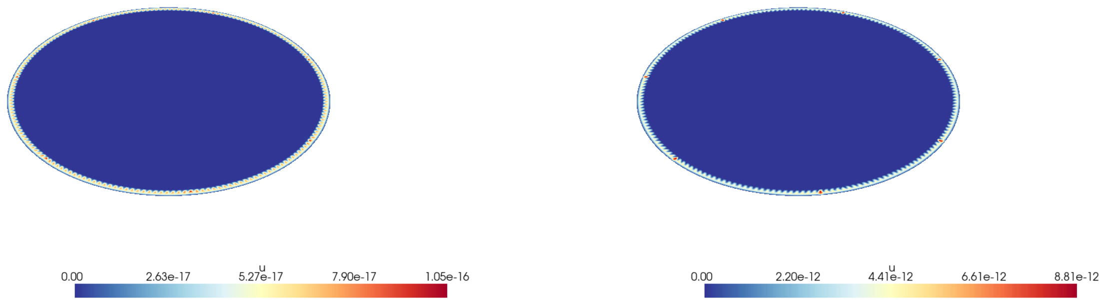


Figure 76: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 5 using order 8 quadrature.

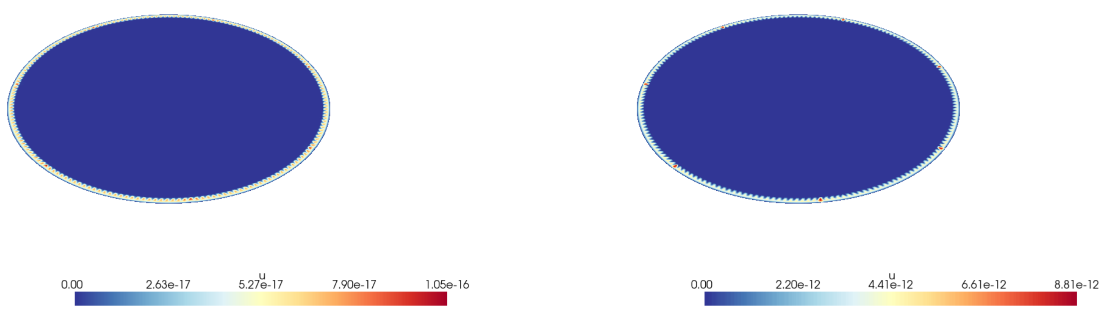


Figure 77: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 5 using order 13 quadrature.

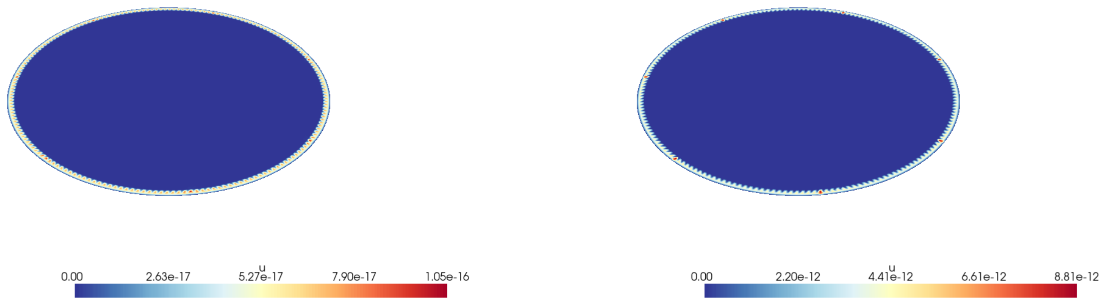


Figure 78: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 5 using order 19 quadrature.

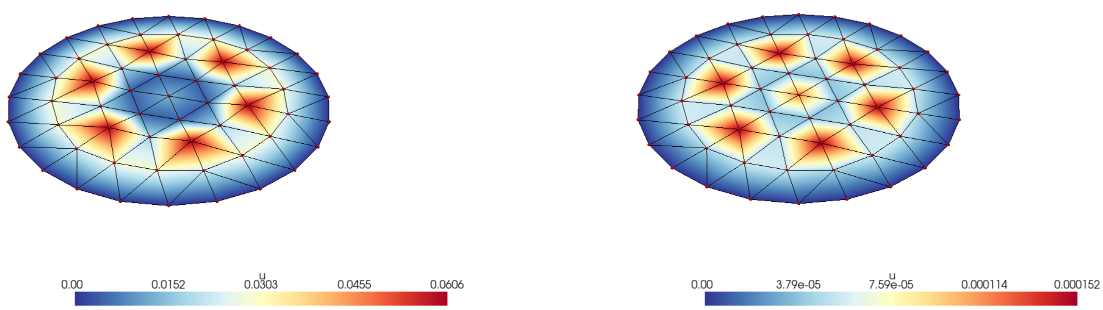


Figure 79: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 1 using order 2 quadrature.

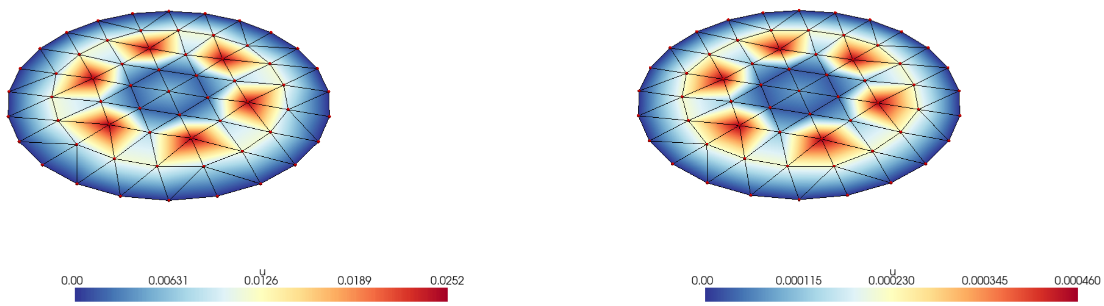


Figure 80: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 1 using order 5 quadrature.

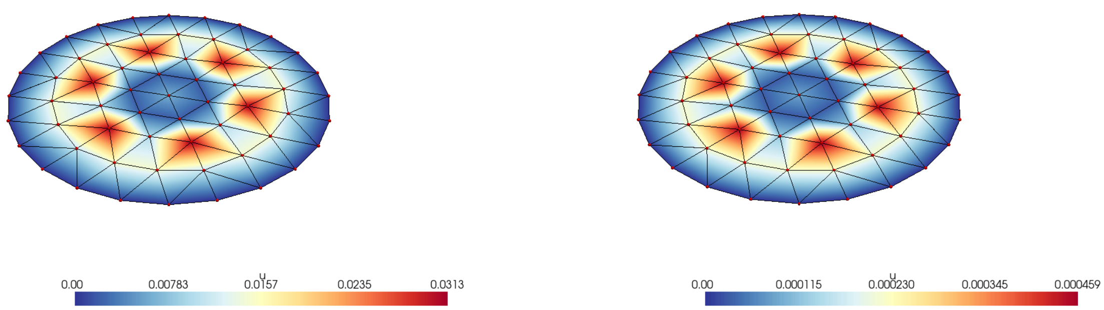


Figure 81: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 1 using order 8 quadrature.

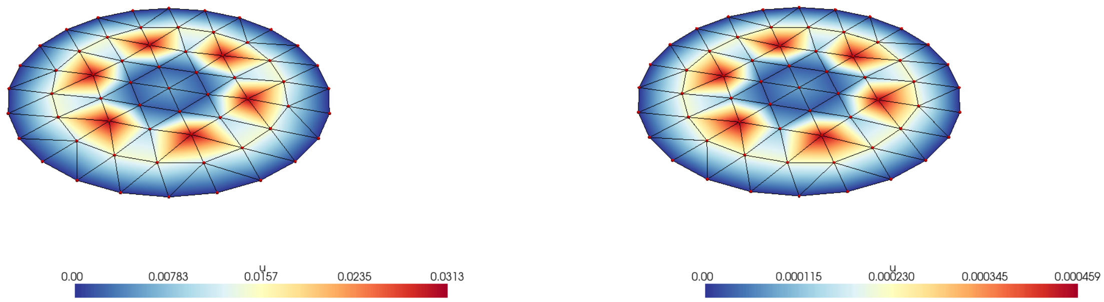


Figure 82: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 1 using order 13 quadrature.

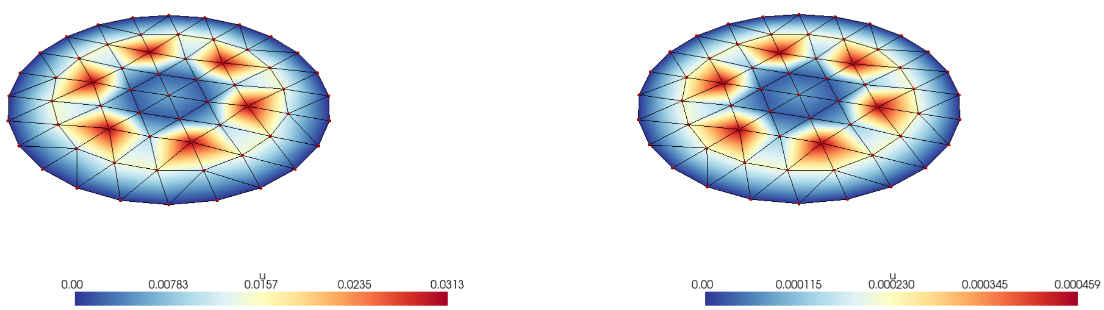


Figure 83: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 1 using order 19 quadrature.

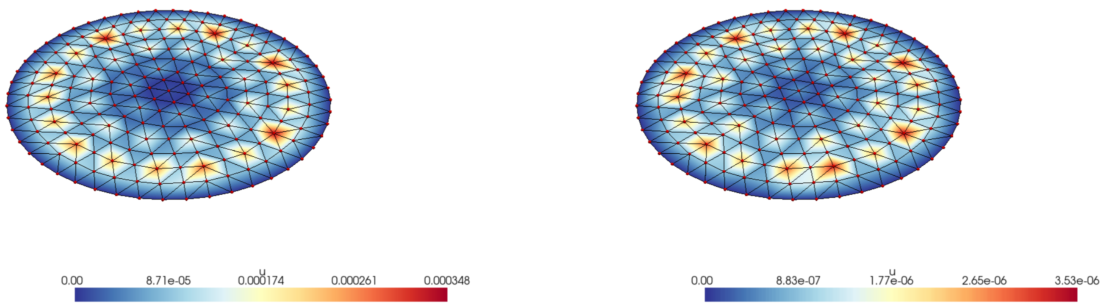


Figure 84: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 2 using order 2 quadrature.

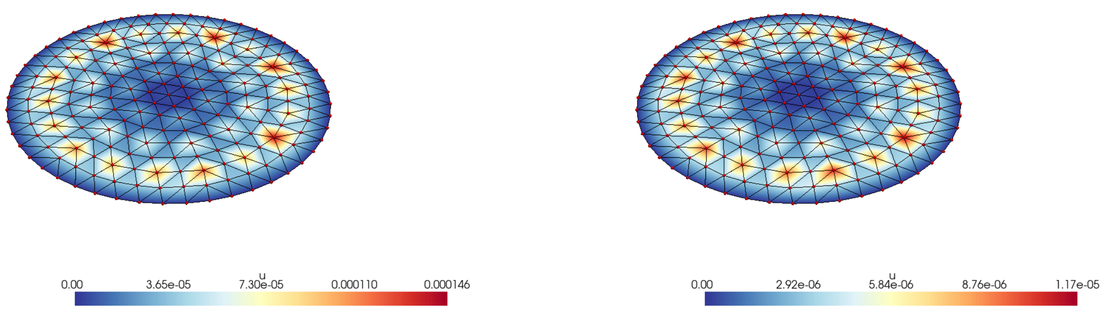


Figure 85: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 2 using order 5 quadrature.

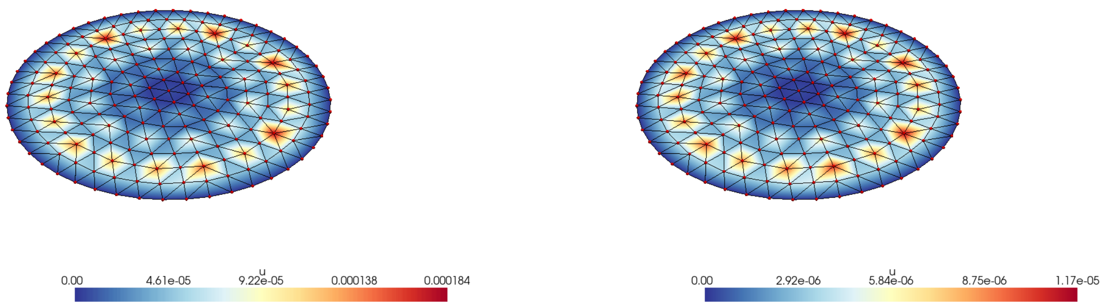


Figure 86: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 2 using order 8 quadrature.

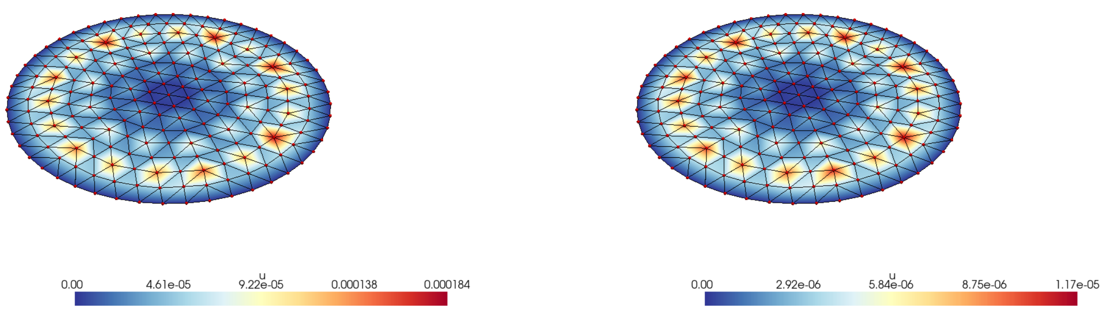


Figure 87: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 2 using order 13 quadrature.

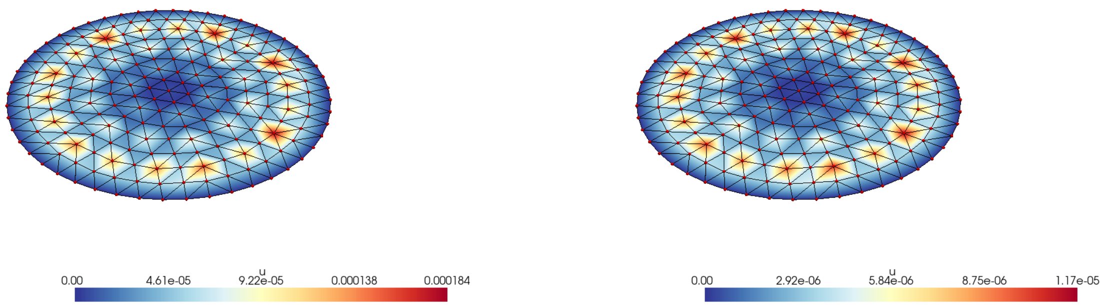


Figure 88: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 2 using order 19 quadrature.

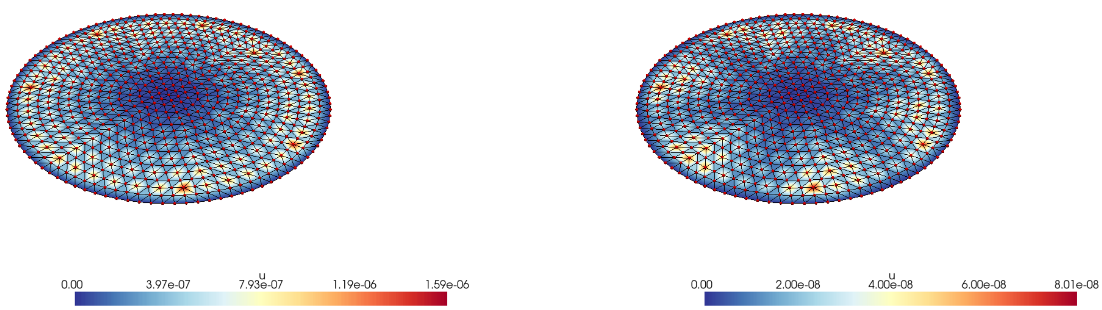


Figure 89: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 3 using order 2 quadrature.

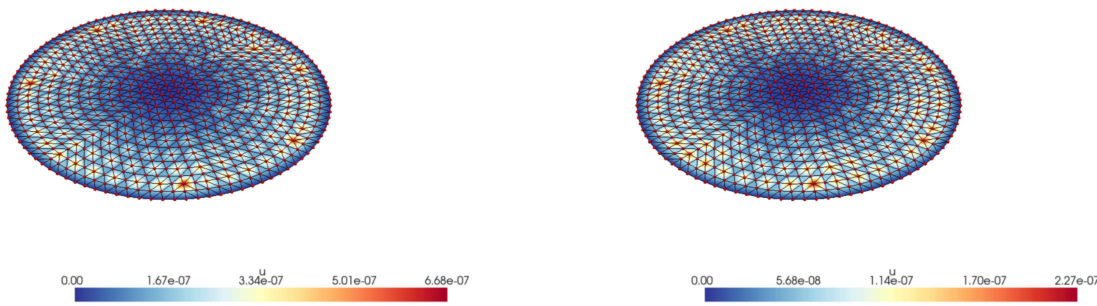


Figure 90: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 3 using order 5 quadrature.

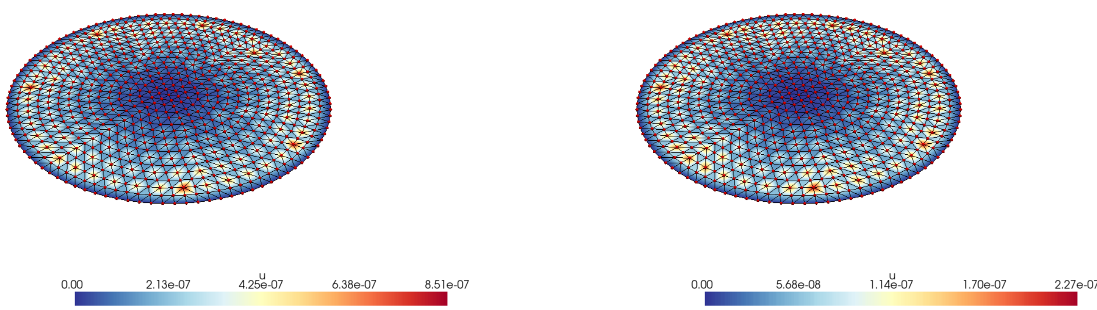


Figure 91: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 3 using order 8 quadrature.

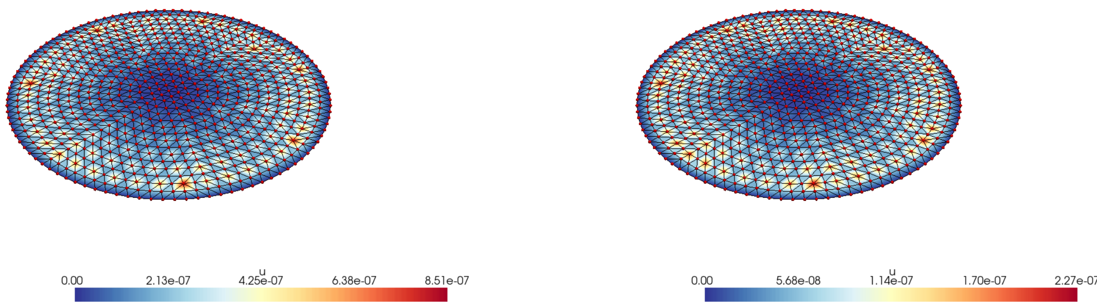


Figure 92: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 3 using order 13 quadrature.

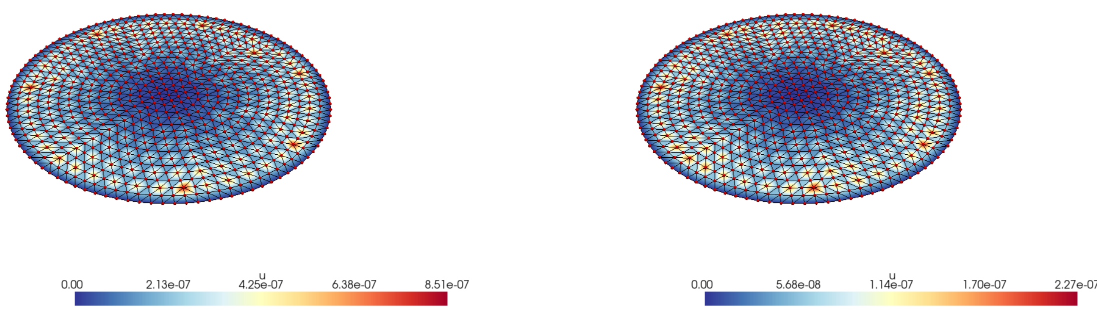


Figure 93: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 3 using order 19 quadrature.

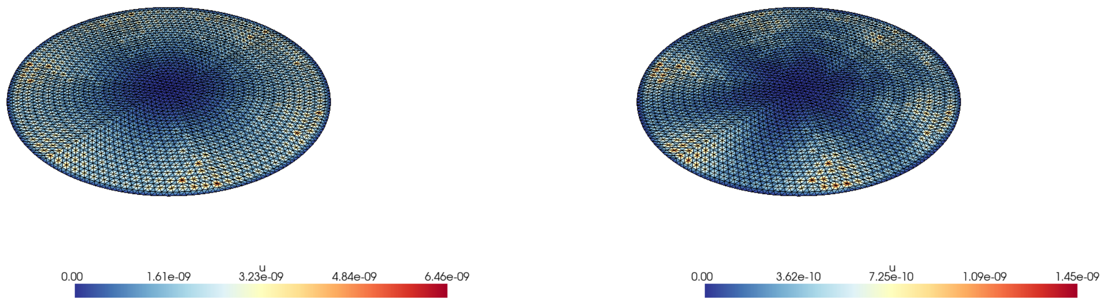


Figure 94: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 4 using order 2 quadrature.

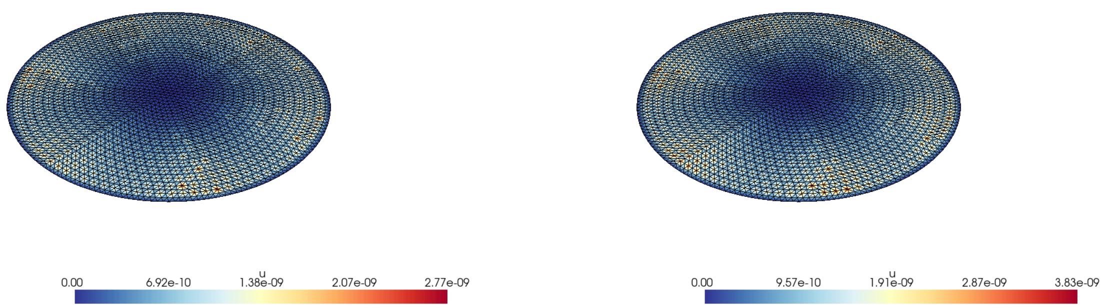


Figure 95: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 4 using order 5 quadrature.

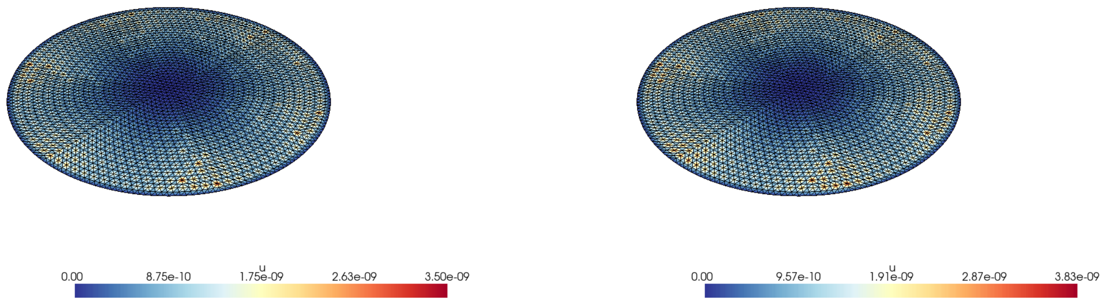


Figure 96: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 4 using order 8 quadrature.

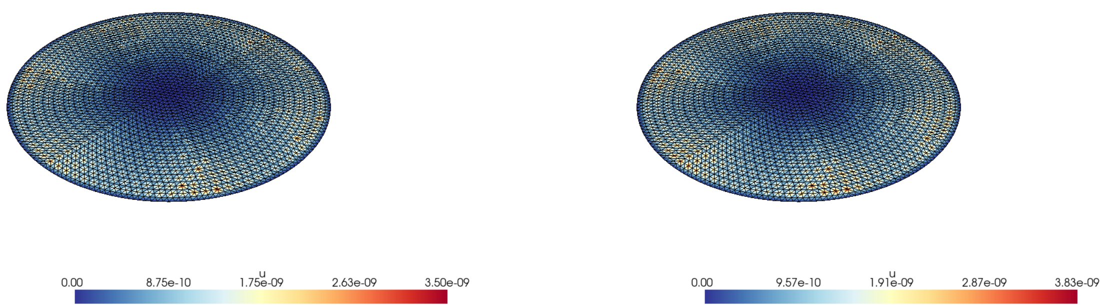


Figure 97: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 4 using order 13 quadrature.

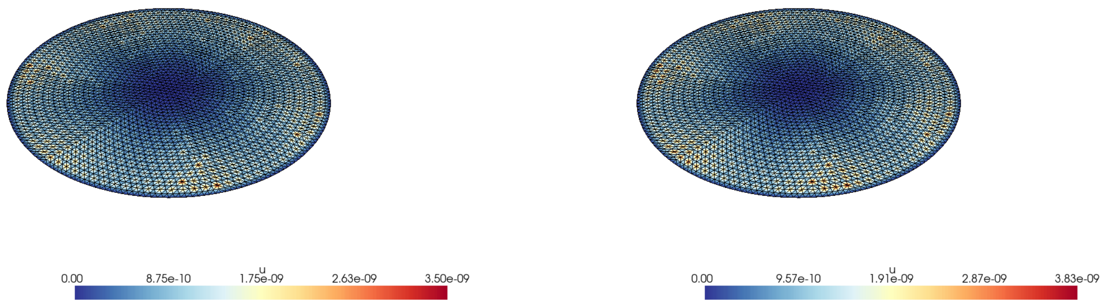


Figure 98: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 4 using order 19 quadrature.

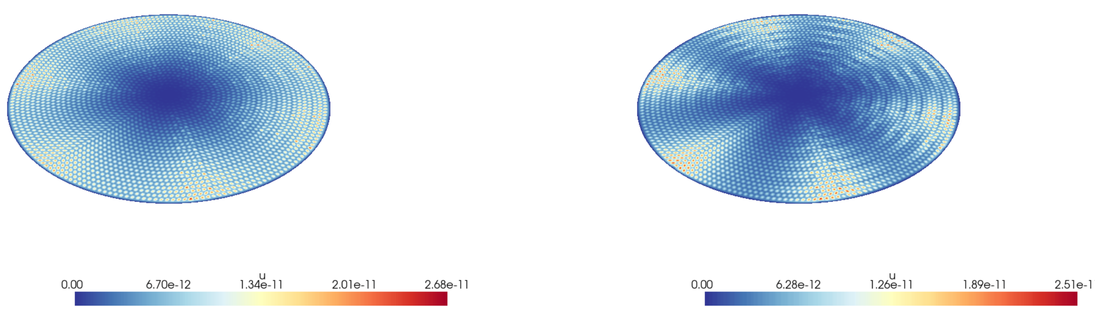


Figure 99: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 5 using order 2 quadrature.

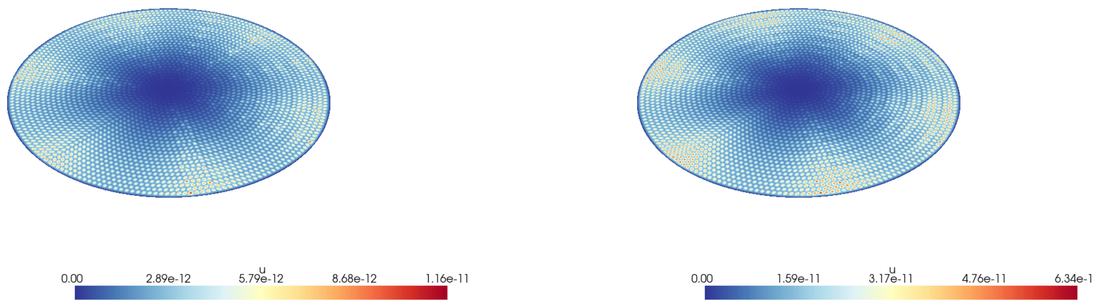


Figure 100: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 5 using order 5 quadrature.

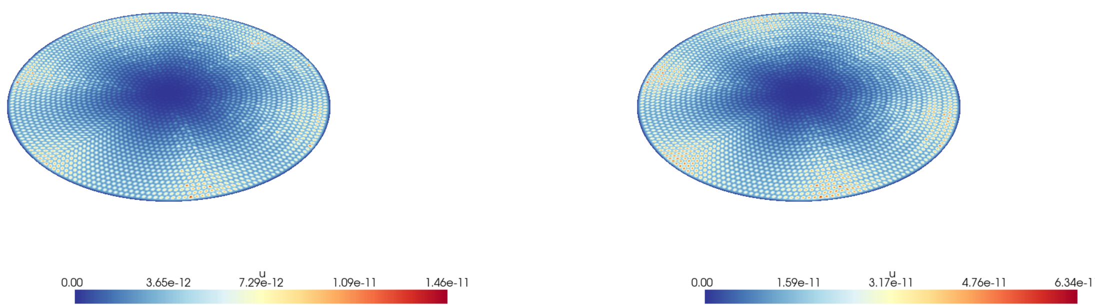


Figure 101: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 5 using order 8 quadrature.

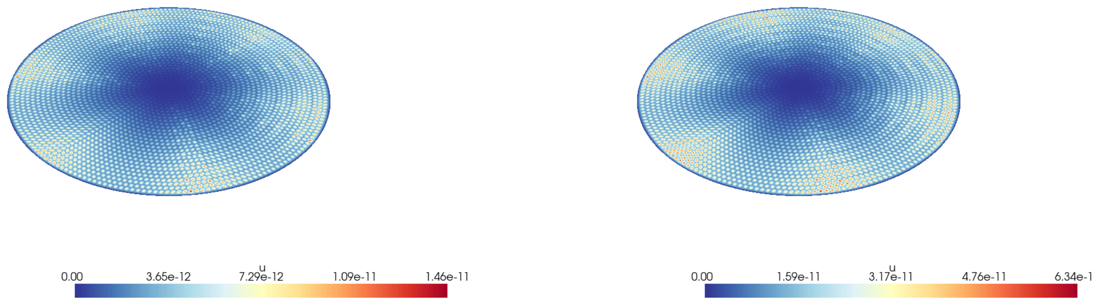


Figure 102: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 5 using order 13 quadrature.

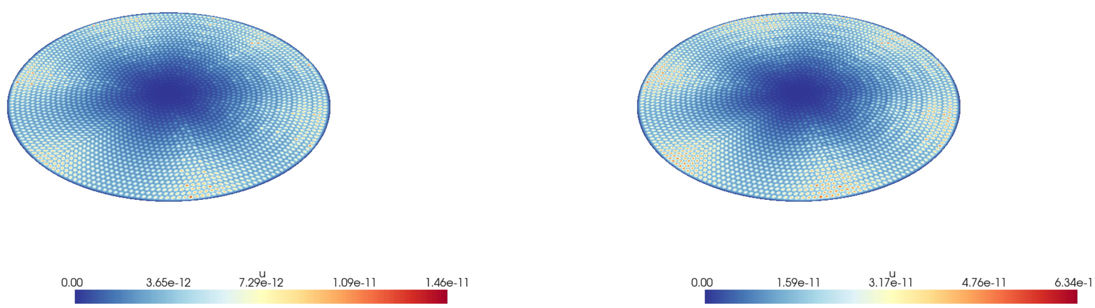


Figure 103: Finite element error in the L2 and H1 norms/seminorms, respectively for problem 1 over mesh number 5 using order 19 quadrature.

Source Code of Interest

```
# Claudio Perez
# May 2021
import jax
import anon.diff as diff
from anabel.template import template
import anabel.backend as anp


@template(6)
def poisson2(transf, test, trial, f=lambda x: 0.0, ndim=2, points=None, weights=None, thickness
    """
    Parameters
    ----------
    test, trial : Callable
        test and trial interpolants over the reference element.
    thickness : float

    http://people.inf.ethz.ch/arbenz/FEM17/pdfs/0-19-852868-X.pdf
    """
    state = {}

    det = anp.linalg.det
    slv = anp.linalg.solve

    jacn_test = diff.jacx(test)
    jacn_trial = diff.jacx(trial)

    def transf(xi, xyz):
        return test(xi)@xyz

    def jacn_transf(xi,xyz):
        return jacn_test(xi)@xyz

    def jacx_test(xi,xyz):
        return slv(jacn_transf(xi,xyz), jacn_test(xi))

    def dvol(xi, xyz):
        return 0.5*thickness*(abs(det(jacn_transf(xi,xyz))))

    def stif(u,xyz,xi,wght,**kwds):
        dNdx = jacx_test(xi,xyz)
        return (dNdx.T@dNdx)*dvol(xi,xyz)*wght

    fj = jax.vmap(f,0)

    def resp(u,xyz,xi,wght,**kwds):
```

```
        dNdx = jacx_test(xi,xyz)
        N = test(xi)[:,None]
        p = (dNdx.T@dNdx)@u*dvol(xi,xyz)*wght - (N@N.T)@fj(xyz)[:,None]*dvol(xi,xyz)*wght
        return p

    integral = jax.vmap(resp,(None,None,0,0))
    jac_integral = jax.vmap(stif,(None, None, 0, 0))

    def jacx(u,__,___,xyz,points,weights):
        return sum(jac_integral(u,xyz,points,weights))

    def main(u,__,___,xyz,points,weights):
        return sum(integral(u,xyz,points,weights))
    return locals()

@template(1)
def L2(transf,test,trial,u,quad_point=None, thickness=1.0):
    state = None
    det = anp.linalg.det
    slv = anp.linalg.solve
    du = lambda x: diff.jacfwd(u)(x)[:,None]
    jacn_test = diff.jacx(test)
    jacn_trial = diff.jacx(trial)

    def transf(xi, xyz):
        return test(xi)@xyz

    def jacn_transf(xi,xyz):
        return jacn_test(xi)@xyz

    dvol = lambda xi, xyz: 0.5*thickness*abs(det(jacn_transf(xi,xyz)))

    def resp(U,xyz,xi, wght):
        N = test(xi)[:,None]
        tmp = u(transf(xi,xyz)) - N.T@U
        q =  tmp.T@tmp * dvol(xi,xyz) * wght
        return q

    integral = jax.vmap(resp,(None,None,0,0))

    def main(u,__,___,xyz,points,weights):
        return sum(integral(u,xyz,points,weights))

    return locals()


@template(1)
def H1_v1(transf,test,trial,u,quad_point=None, thickness=1.0):
```

```
    state = None
    det = anp.linalg.det
    slv = anp.linalg.solve
    du = lambda x: diff.jacfwd(u)(x)[:,None]
    jacn_test = diff.jacx(test)
    jacn_trial = diff.jacx(trial)

    def transf(xi, xyz):
        return test(xi)@xyz

    def jacn_transf(xi,xyz):
        return jacn_test(xi)@xyz

    jacx_test = lambda xi,xyz: slv(jacn_transf(xi,xyz), jacn_test(xi))
    dvol = lambda xi, xyz: 0.5*thickness*abs(det(jacn_transf(xi,xyz)))


    def resp(U,xyz,xi, wght):
        tmp = du(transf(xi,xyz)) - jacx_test(xi,xyz)@U
        q = tmp.T@tmp * dvol(xi,xyz) * wght
        return q

    integral = jax.vmap(resp,(None,None,0,0))

    def main(u,__,___,xyz,points,weights):
        return sum(integral(u,xyz,points,weights))

    return locals()

@template(1)
def H1(transf,test,trial,u,quad_point=None, thickness=1.0):
    state = None
    det = anp.linalg.det
    slv = anp.linalg.solve
    du = lambda x: diff.jacfwd(u)(x)[:,None]
    jacn_test = diff.jacx(test)
    jacn_trial = diff.jacx(trial)

    def transf(xi, xyz):
        return test(xi)@xyz

    def jacn_transf(xi,xyz):
        return jacn_test(xi)@xyz

    jacx_test = lambda xi,xyz: slv(jacn_transf(xi,xyz), jacn_test(xi))
    dvol = lambda xi, xyz: 0.5*thickness*abs(det(jacn_transf(xi,xyz)))

    def resp(U,xyz,xi, wght):
```

```
        tmp = jacx_test(xi,xyz)@(U - u(transf(xi,xyz)))
        q = tmp.T@tmp * dvol(xi,xyz) * wght
        return q

    integral = jax.vmap(resp,(None,None,0,0))

    def main(u,__,___,xyz,points,weights):
        return sum(integral(u,xyz,points,weights))

    return locals()
```